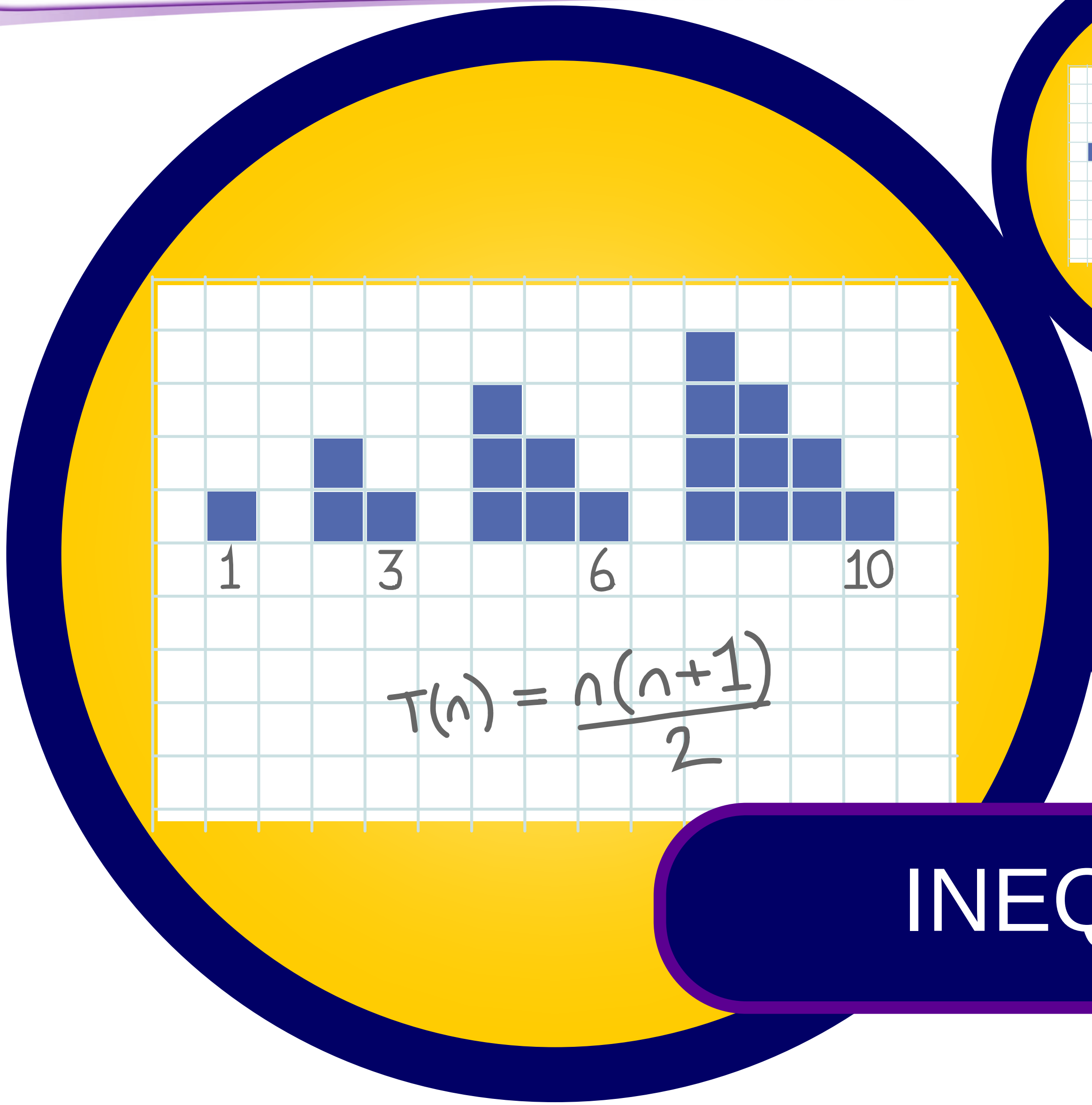
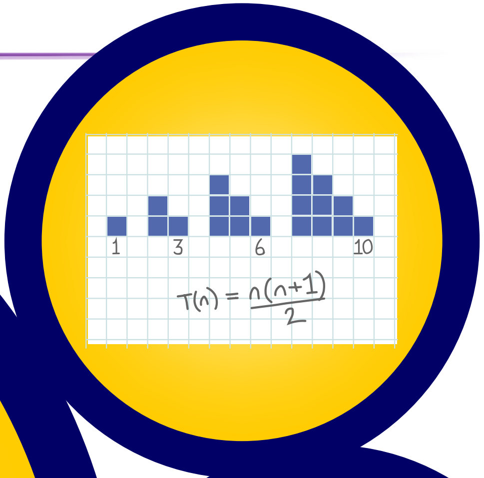
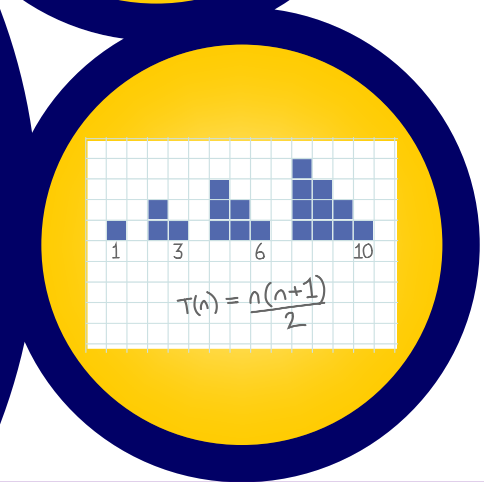






INEQUALITIES

Contents

A4 Inequalities

Inequalities

An **inequality** is an algebraic statement involving the symbols

$$>, <, \geq \text{ or } \leq$$

For example,

$x > 3$	means	'x is **greater than** 3'.
$x < -6$	means	'x is **less than** –6'.
$x \geq -2$	means	'x is **greater than or equal to** –2'.
$x \leq 10$	means	'x is **less than or equal to** 10'.

Sometimes two inequalities can be combined in a single statement. For example,

If $x > 3$ **and** $x \leq 14$ we can write

$$3 < x \leq 14$$

Reversing inequalities

Inequalities can either be read from left to right or from right to left. For example,

$$5 > -3$$

can be read as '5 is greater than –3' by reading from left to right.

It can also be read as '–3 is less than 5' by reading from right to left.

In general,

$x > y$ is equivalent to $y < x$

and

$x \geq y$ is equivalent to $y \leq x$

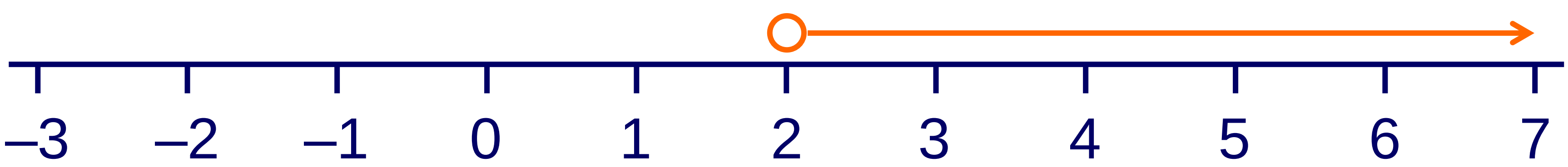
Representing inequalities on number lines

Suppose $x > 2$. There are infinitely many values that x could have.

x could be equal to 3, 7.3, $54\frac{3}{11}$, 18463.431 …

It would be impossible to write every solution down.

We can therefore represent the **solution set** on a number line as follows:



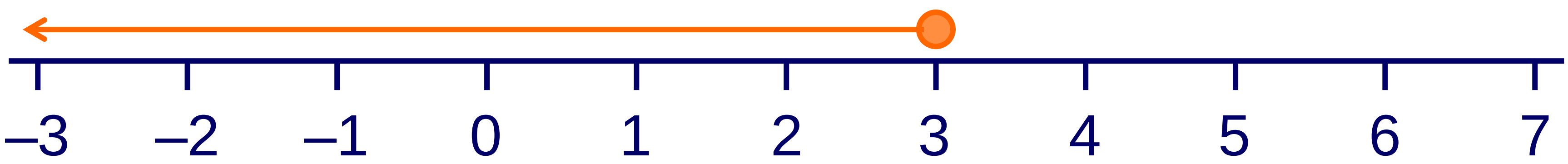
A hollow circle, ○, at 2 means that this number is not included and the arrow at the end of the line means that the solution set extends in the direction shown.

Representing inequalities on number lines

Suppose $x \leq 3$. Again, there are infinitely many values that x could have.

x could be equal to 3, –1.4, $-94\frac{8}{17}$, –7452.802 …

We can represent the **solution set** on a number line as follows,



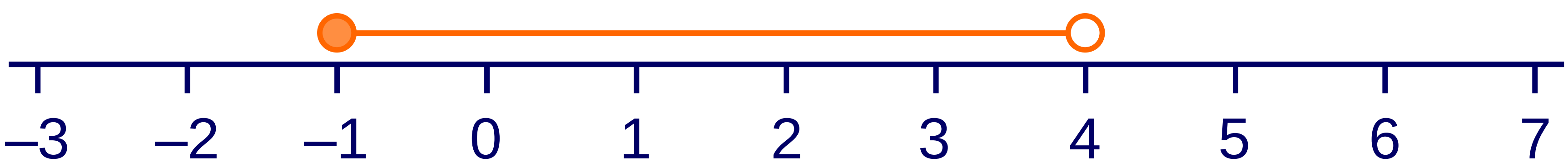
A solid circle, ●, at 3 means that this number is included and the arrow at the end of the line means that the solution set extends in the direction shown.

Representing inequalities on number lines

Suppose $-1 \leq x < 4$. Although x is between two values, there are still infinitely many values that x could have.

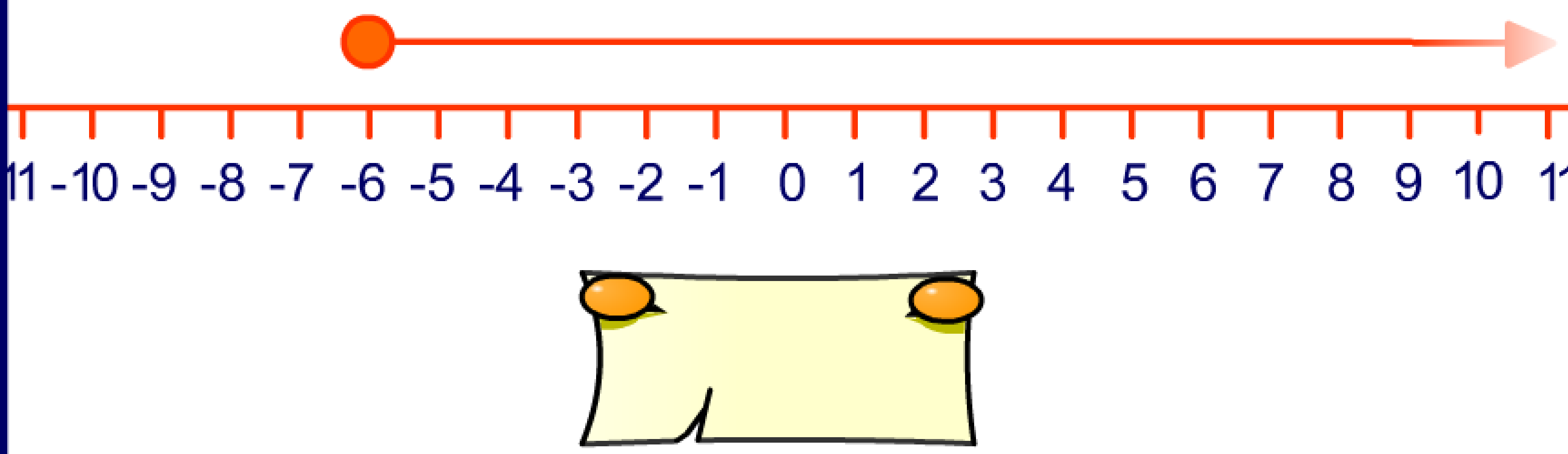
x could be equal to 2, -0.7, $-3\frac{16}{17}$, 1.648953 …

We can represent the **solution set** on a number line as follows:



A solid circle, ●, is used at –1 because this value is included and a hollow circle, ○, is used at 4 because this value is not included. The line represents all the values in between.

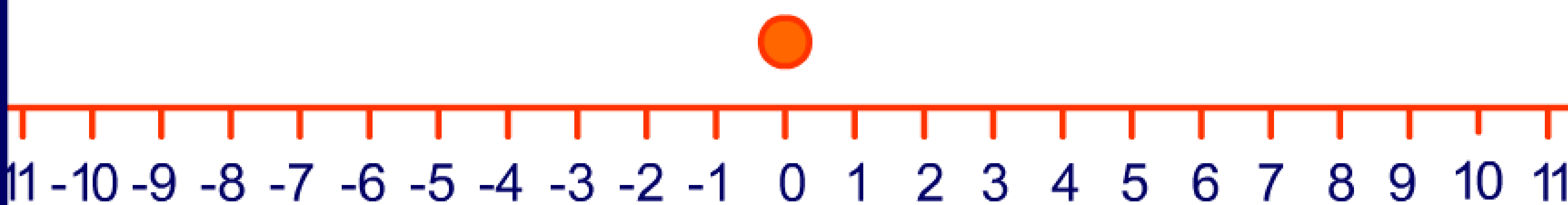
Writing inequalities from number lines

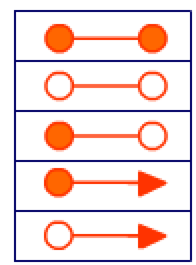


Representing inequalities on number lines

Represent this inequality on the number line:

$-5 > x \geq -7$





Integer solutions

In the examples that we have looked at so far we have assumed that the value of x can be any real number.

Sometimes we are told that x can only be an integer, that is a positive or negative whole number.

For example,

$$-3 < x \leq 5$$

List the integer values that satisfy this inequality.

The integer values that satisfy this inequality are

–2, –1, 0, 1, 2, 3, 4, 5.

Integer solutions

Write down an inequality that is obeyed by the following set of integers:

$$-4,\ -3,\ -2,\ -1,\ 0,\ 1.$$

There are four possible inequalities that give this solution set,

$$-5 < x < 2$$

$$-4 \leq x < 2$$

$$-5 < x \leq 1$$

$$-4 \leq x \leq 1$$

Remember that when we use < and > the values at either end are not included in the solution set.

Contents

A4 Inequalities

Solving linear inequalities

Look at the following inequality,

$$x + 3 \geq 7$$

What values of x would make this inequality true?

Any value of x greater or equal to 4 would **solve** this inequality.

We could have solved this inequality as follows,

$$x + 3 \geq 7$$

subtract 3 from both sides: $x + 3 - 3 \geq 7 - 3$

$$x \geq 4$$

The solution has one letter on one side of the inequality sign and a number on the other.

Solving linear inequalities

Like an equation, we can solve an inequality by adding or subtracting the same value to both sides of the inequality sign.

We can also multiply or divide both sides of the inequality by a **positive** value. For example,

Solve $4x - 7 > 11 - 2x$

add 7 to both sides: $4x > 18 - 2x$

add $2x$ to both sides: $6x > 18$

divide both sides by 6: $x > 3$

How could we check this solution?

Checking solutions

To verify that $x > 3$

is the solution to $4x - 7 > 11 - 2x$

substitute a value just above 3 into the inequality and then substitute a value just below 3.

If we substitute $x = 4$ into the inequality we have

$$4 \times 4 - 7 > 11 - 2 \times 4$$

$$16 - 7 > 11 - 8$$

$$9 > 3$$ This is true.

If we substitute $x = 2$ into the inequality we have,

$$4 \times 2 - 7 > 11 - 2 \times 2$$

$$8 - 7 > 11 - 4$$

$$1 > 7$$ This is not true.

Multiplying or dividing by negatives

Although most inequalities can be solved like equations we have to take great care when multiplying or dividing both sides of an inequality by a negative value.

The following simple inequality is true,

$$-3 < 5$$

Look what happen if we multiply both sides by –1,

$$-3 \times -1 < 5 \times -1$$

~~3 < –5~~

3 is not less than –5. To keep the inequality true we have to reverse the inequality sign.

$$3 > -5$$ ✓

Multiplying or dividing by negatives

Remember, when both sides of an inequality are multiplied or divided by a negative number the inequality is reversed.

For example, $4 - 3x \leq 10$

subtract 4 from both sides: $-3x \leq 6$

divide both side by –3: $x \geq -2$ The inequality sign is reversed.

We could also solve this type of inequality by collecting x terms on the right and reversing the inequality sign at the end.

$$4 - 3x \leq 10$$

add $3x$ to both sides: $4 \leq 10 + 3x$

subtract 10 from both sides: $-6 \leq 3x$

divide both sides by 3: $-2 \leq x$

$$x \geq -2$$

Solving combined linear inequalities

The two inequalities $4x + 3 \geq 5$ and $4x + 3 < 15$ can be written as a single combined inequality.

$$5 \leq 4x + 3 < 15$$

We can solve this inequality as follows:

subtract 3 from each part: $2 \leq 4x < 12$

divide each part by 4: $0.5 \leq x < 3$

We can illustrate this solution on a number line as

Solving combined linear inequalities

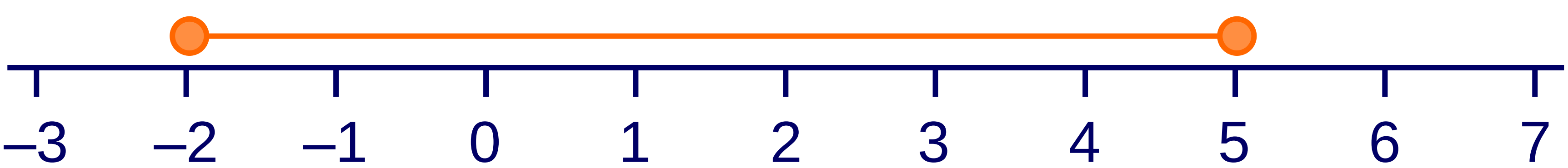
Some combined inequalities contain variables in more than one part. For example,

$$x - 2 \leq 3x + 2 \leq 2x + 7$$

Treat this as two separate inequalities,

$x - 2 \leq 3x + 2$	and	$3x + 2 \leq 2x + 7$
$-2 \leq 2x + 2$		$x + 2 \leq 7$
$-4 \leq 2x$		$x \leq 5$
$-2 \leq x$		

We can write the complete solution as $-2 \leq x \leq 5$ and illustrate it on a number line as:



Overlapping solutions

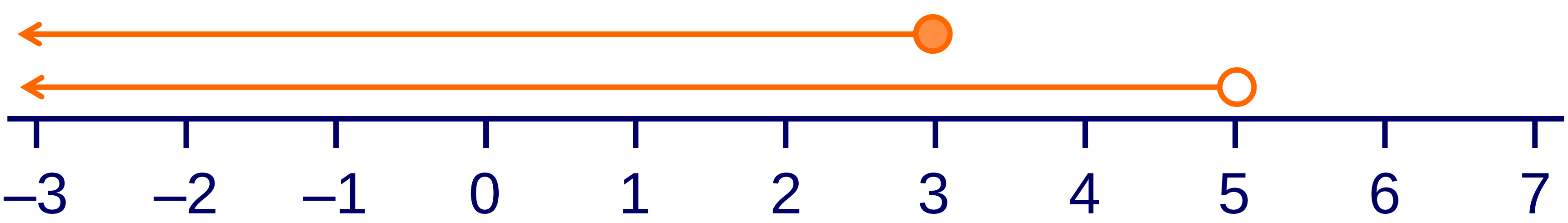
Solve the following inequality and illustrate the solution on a number line.

$$2x - 1 \leq x + 2 < 7$$

Treating as two separate inequalities,

$2x - 1 \leq x + 2$	and	$x + 2 < 7$
$x - 1 \leq 2$		$x < 5$
$x \leq 3$		

If $x < 5$ then it is also ≤ 3. The whole solution set is therefore given by $x \leq 3$. This is can be seen on the number line:



Solutions in two parts

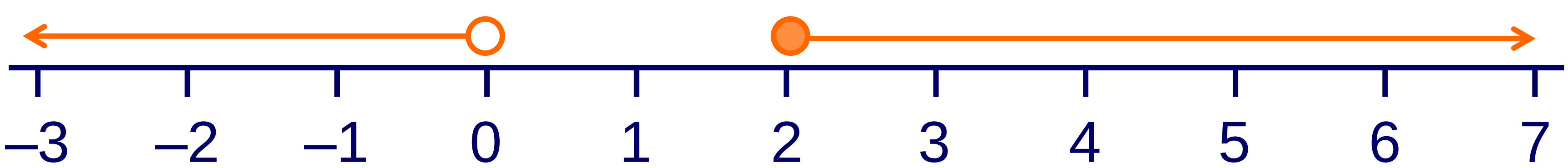
Solve the following inequality and illustrate the solution on a number line:

$$4x + 5 < 3x + 5 \leq 4x + 3$$

Treating as two separate inequalities,

$4x + 5 < 3x + 5$	and	$3x + 5 \leq 4x + 3$
$4x < 3x$		$5 \leq x + 3$
$x < 0$		$2 \leq x$
		$x \geq 2$

We cannot write these solutions as a single combined inequality. The solution has two parts.



Contents

A4 Inequalities

Vertical regions

Inequalities can be represented by **regions** on a graph.

A region is an area where all the points obey a given rule.

Suppose we want to find the region where

$$x > 2$$

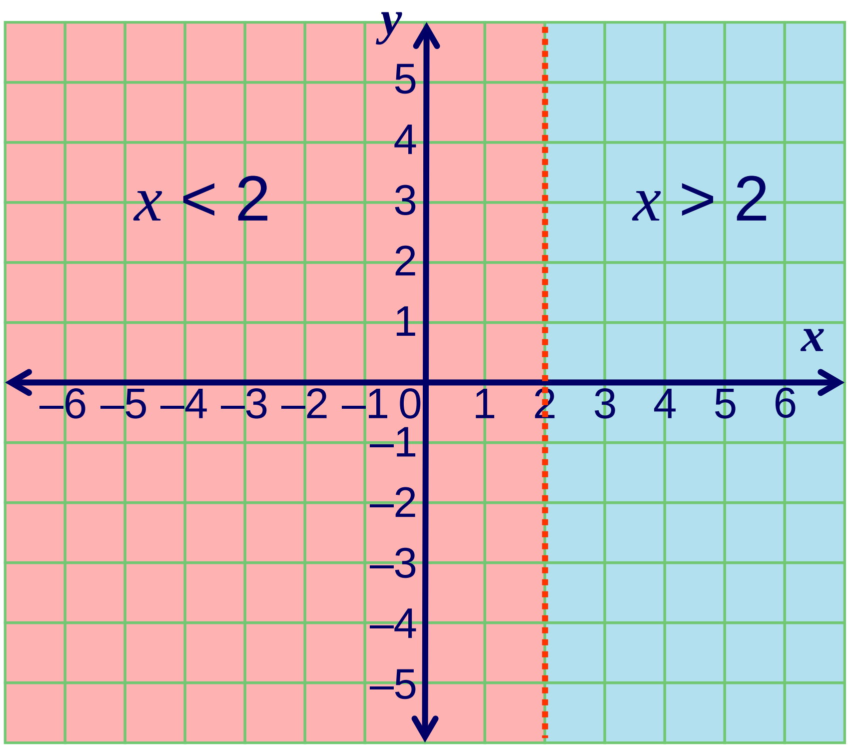
This means that we want to show the area of a graph where the x-coordinate of every point is greater than 2.

Give the coordinates of three points that would satisfy this condition.

For example (4, 1), (6, 5), and (3, –2)

Vertical regions

We can represent all the points where the x-coordinate is equal to 2 with the line $x = 2$.

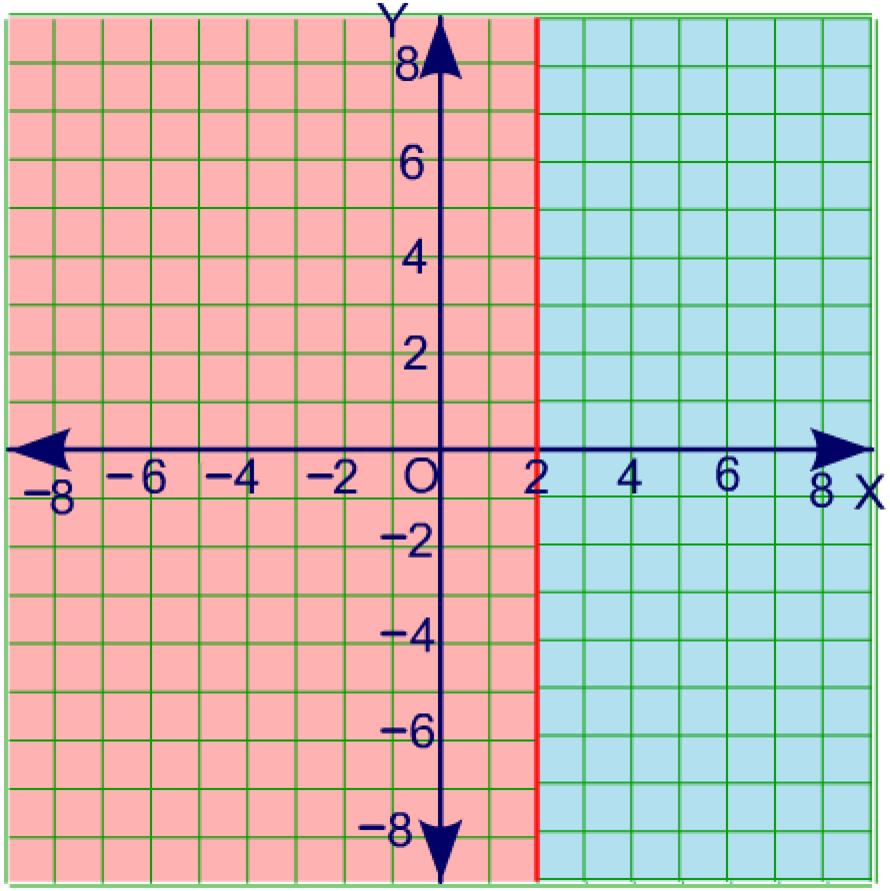


The region where $x > 2$ does not include points where $x = 2$ and so we draw this as a dotted line.

The region to the *right* of the line $x = 2$ contains every point where $x > 2$.

The region to the *left* of the line $x = 2$ contains every point where $x < 2$.

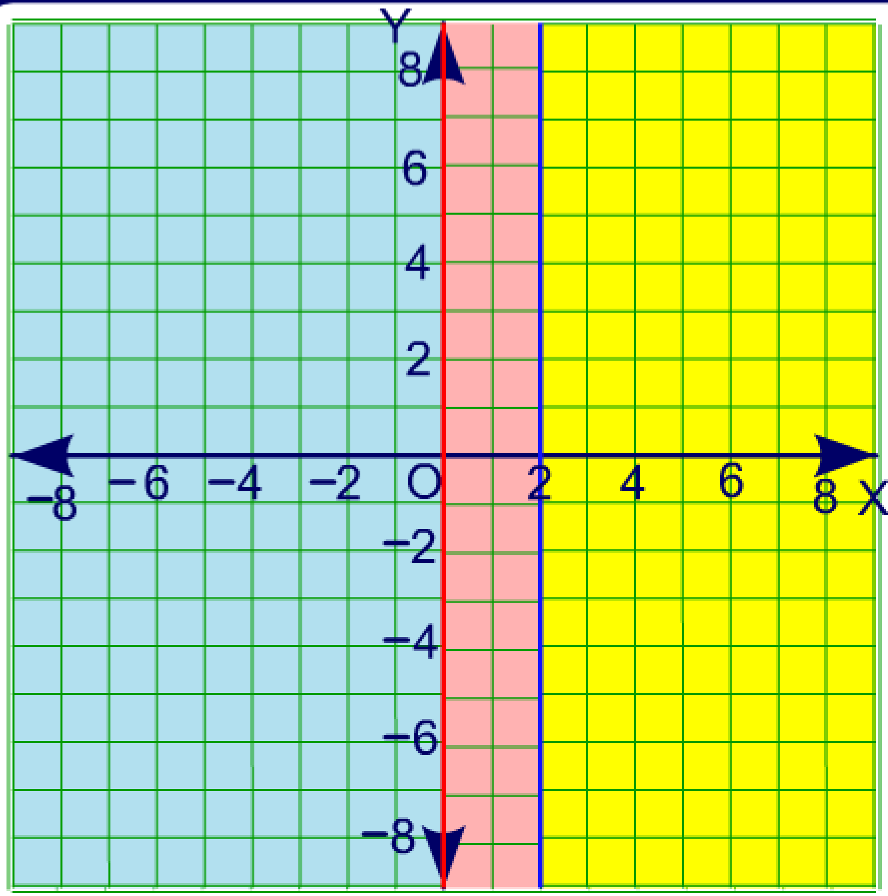
Vertical regions 1



$x \geq 2$

$x \leq 2$

Vertical regions 2



$x \geq 2$

$0 \leq x \leq 2$

$x \leq 0$

Horizontal regions

Suppose we want to find the region where

$$y \leq 3$$

This means that we want to show the area of a graph where the y-coordinate of every point is less than or equal to 3.

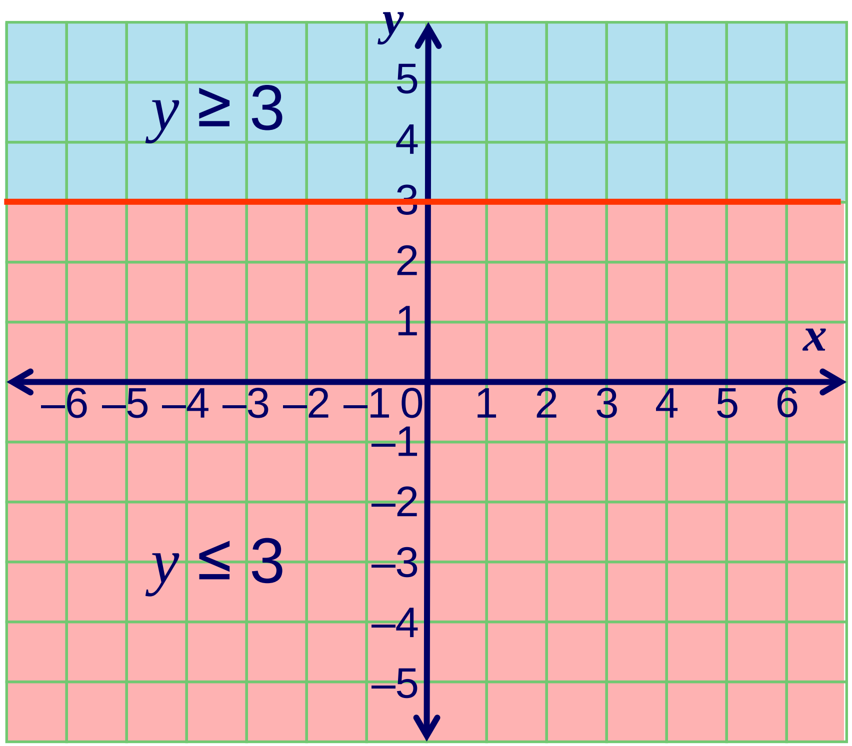
Give the coordinates of three points that would satisfy this condition.

For example, (5, 1), (–3, –4), and (0, 2)

We can represent all the points where the y-coordinate is equal to 3 with the line $y = 3$.

Horizontal regions

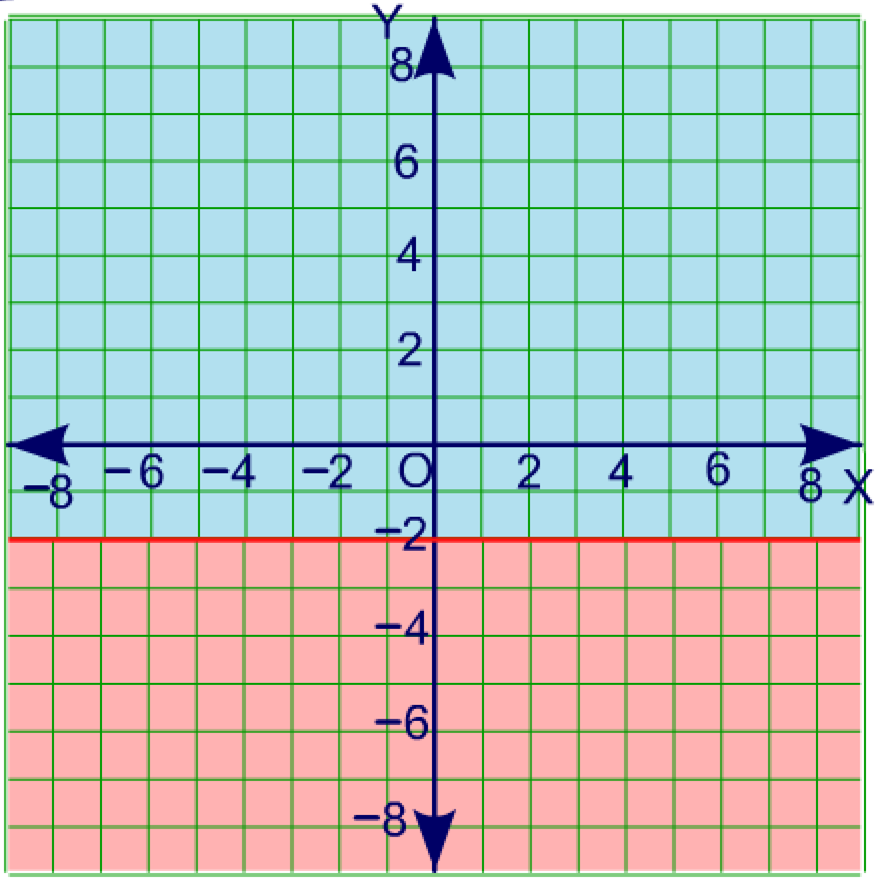
The region where $y \le 3$ includes points where $y = 3$ and so we draw $y = 3$ as a solid line.



The region *below* the line $y = 3$ contains every point where $y \le 3$.

The region *above* the line $y = 3$ contains every point where $y \ge 3$.

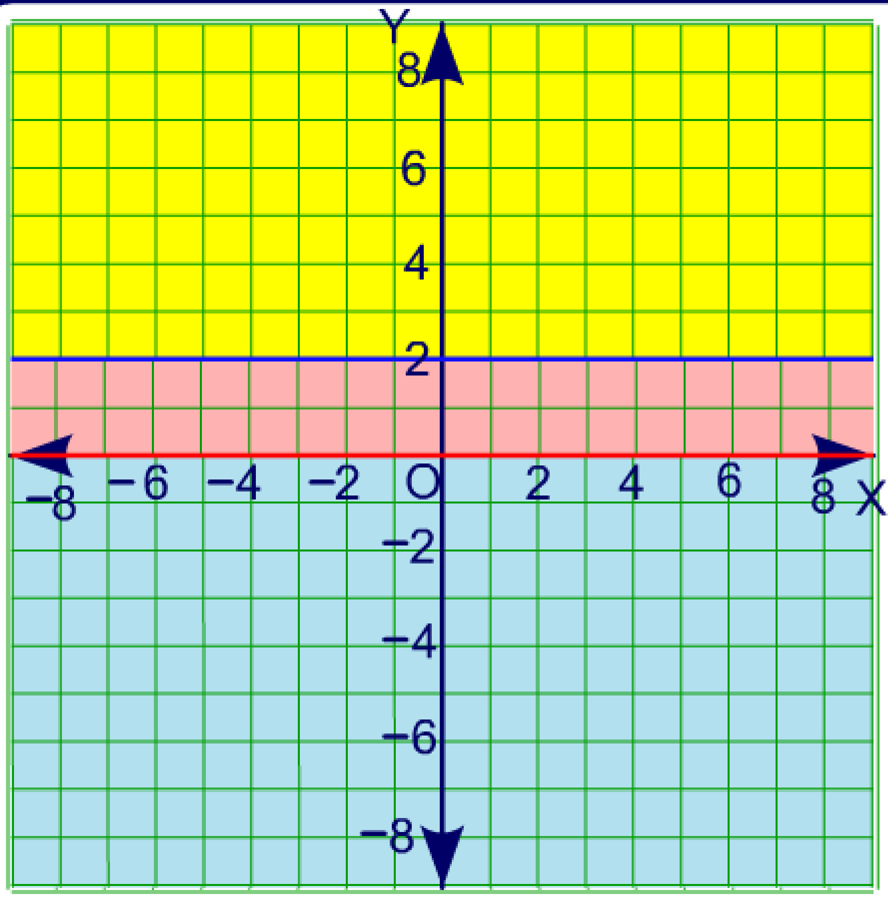
Horizontal regions 1



$y \geq 2$

$y \leq 2$

Horizontal regions 2



$y \geq 2$

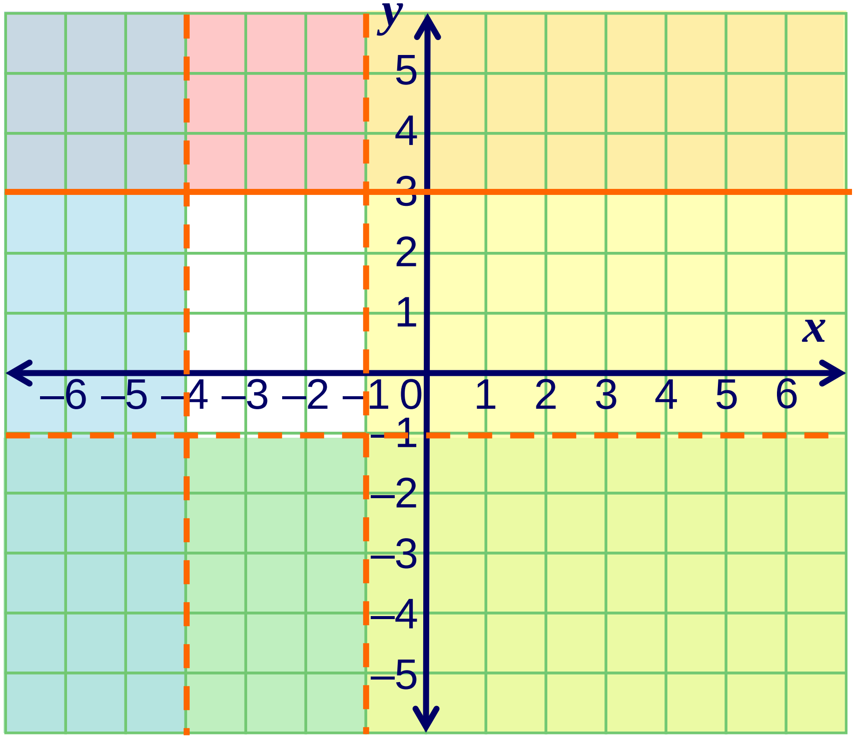
$0 \leq y \leq 2$

$y \leq 0$

Horizontal and vertical regions combined

When several regions are shown on the same graph it is usual to shade out the *unwanted* regions.

This is so that the required area where the regions overlap can easily be identified.



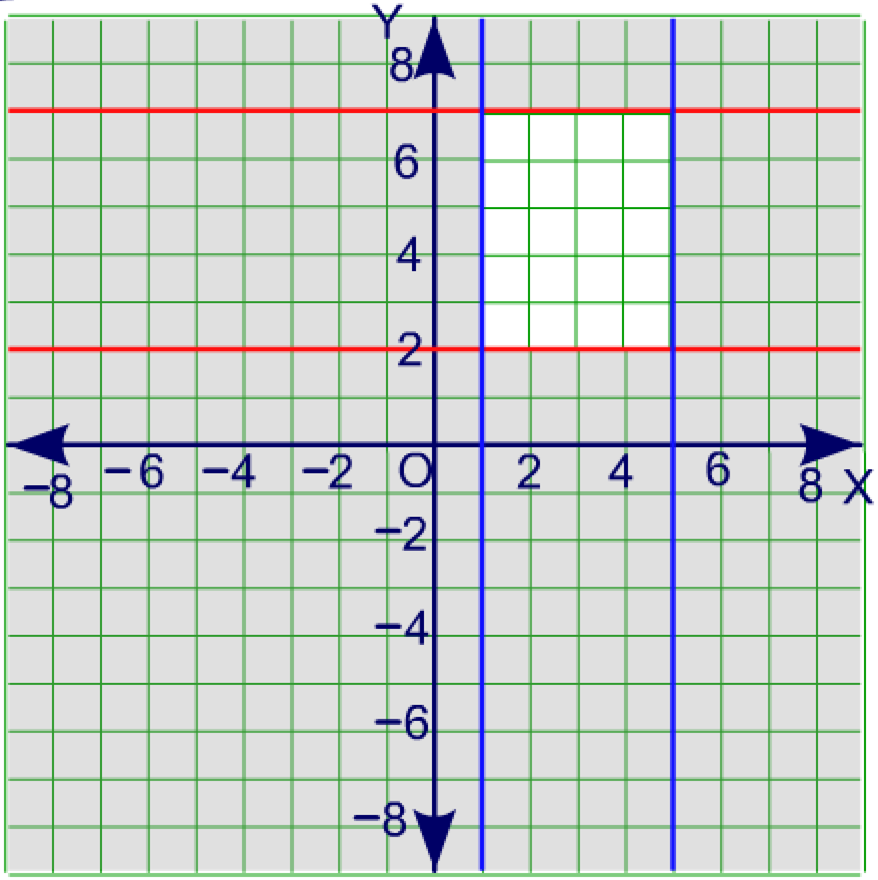
For example, to show the region where $-4 < x < -1$ and $-1 < y \leq 3$,

1) Shade out the regions $x < -4$ and $x > -1$.

2) Shade out the regions $y < -1$ and $y \geq 3$.

The unshaded region satisfies both $-4 < x < -1$ and $-1 < y \leq 3$.

Horizontal and vertical regions combined



$2 \leq y \leq 7$

$1 \leq x \leq 5$

Contents

A4 Inequalities

Inequalities in two variables

Linear inequalities can be given in two variables x and y. For example,

$$x + y < 3$$

The solution set to this inequality is made up of pair of values. For example,

$x = 1$ and $y = 1$

$x = 4$ and $y = –5$

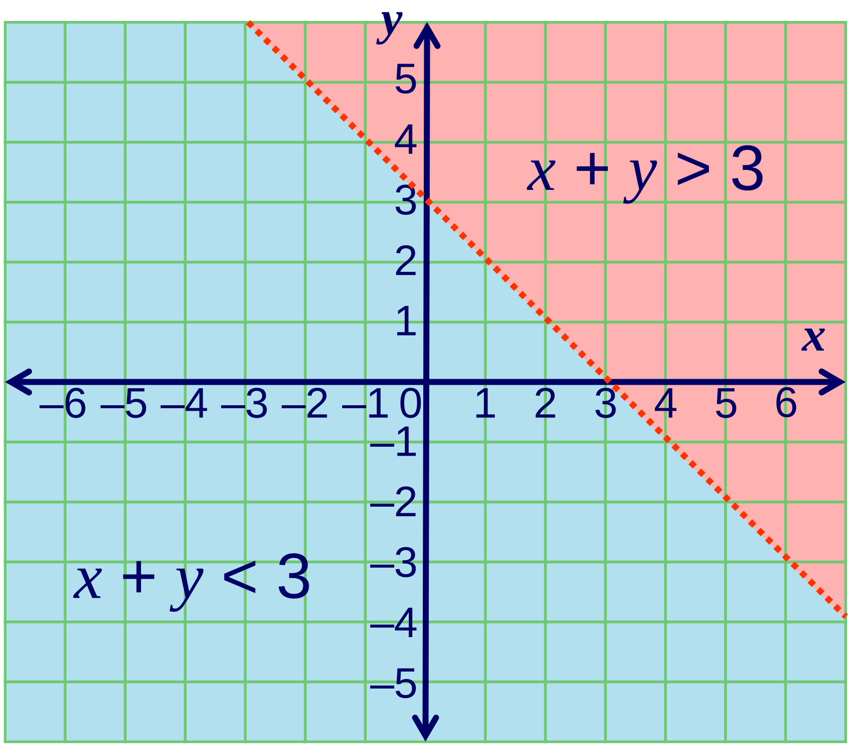
$x = –1$ and $y = 0$

These solutions are usually written as coordinate pairs as (1, 1), (4, –5) and (–1, 0).

The whole solution set can be represented using a graph.

Inequalities in two variables

We can represent all the points where the x-coordinate and the y-coordinate add up to 3 with the line $x + y = 3$.



The region where $x + y < 3$ does not include points where $x + y = 3$ and so we draw this as a dotted line.

The region *below* the line $x + y = 3$ contains every point where $x + y < 3$.

The region *above* the line $x + y = 3$ contains every point where $x + y > 3$.

Inequalities in two variables

When a line is sloping it may not always be obvious which side of the line gives the required region.

We can check this by choosing a point (not on the line) and substituting the x- and y-values of the point into the inequality representing the required region.

If the point satisfies the inequality then it is in the region. If it does not satisfy the inequality it is not in the region.

The easiest point to substitute is usually the point at the origin, that is the point (0, 0).

Inequalities in two variables

For example,

Is the point (0, 0) in the region $4y - 3x > 2$?

Substituting $x = 0$ and $y = 0$ into $4y - 3x > 2$ gives,

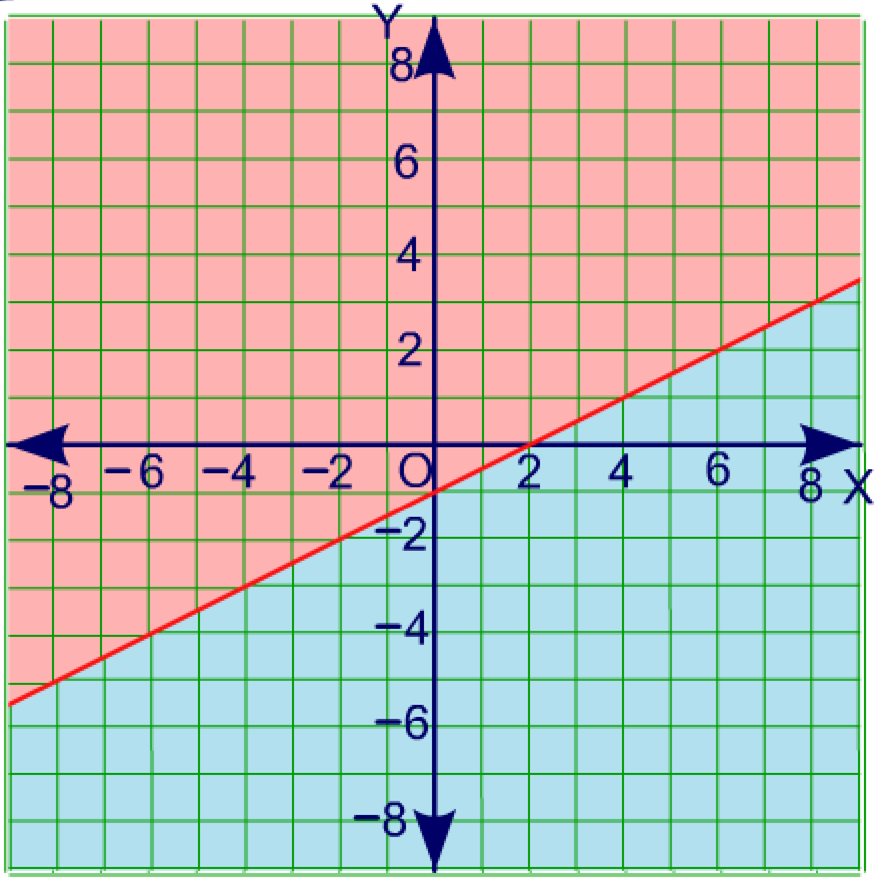
$$4 \times 0 - 3 \times 0 > 2$$

$$0 > 2$$

0 is not greater than 2 and so the point (0, 0) does not lie in the required region.

The region representing $4y - 3x > 2$ is therefore the region that does not contain the point at the origin.

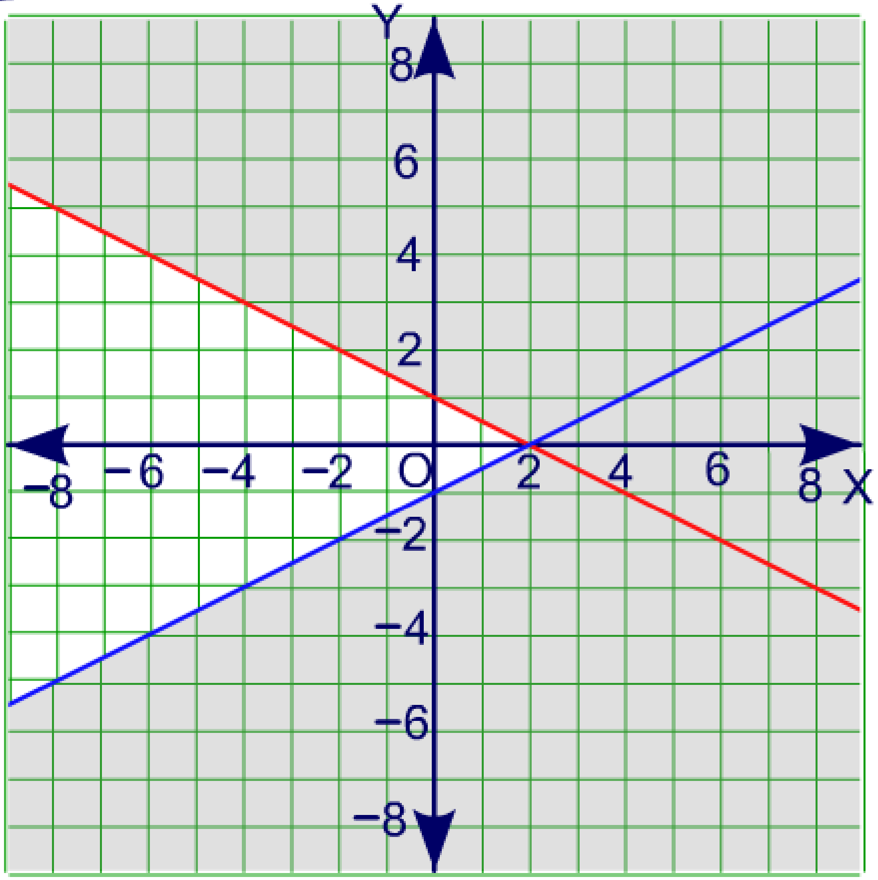
Inequalities in two variables



$x - 2y \geq 2$

$x - 2y \leq 2$

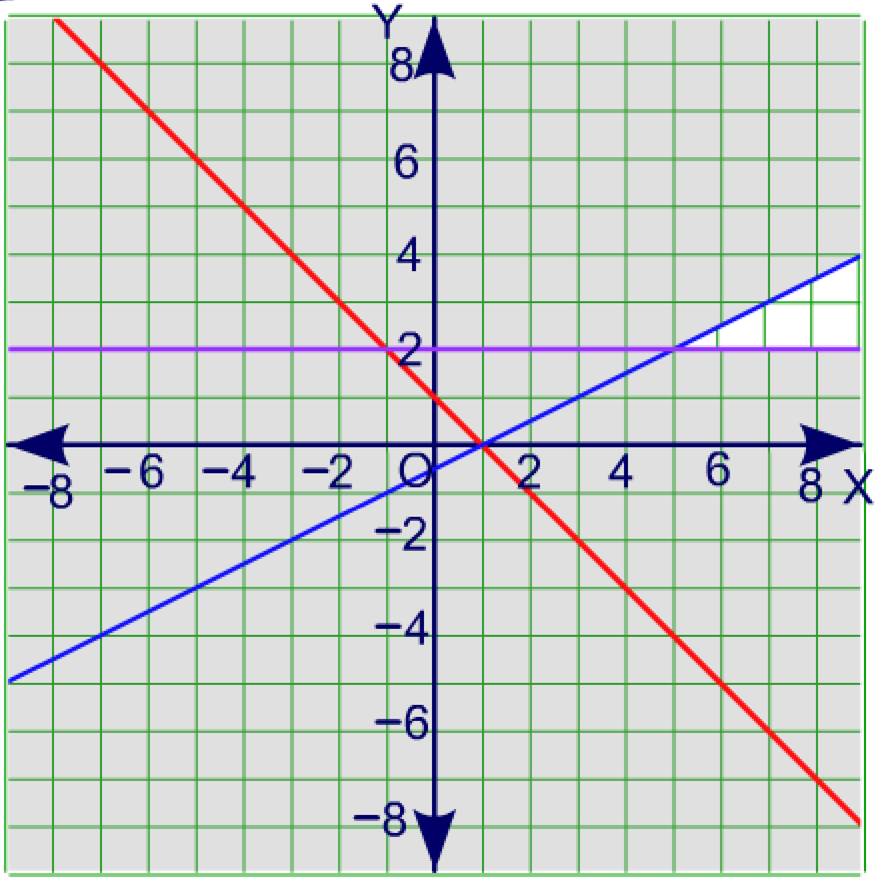
Combining inequalities in two variables 1



$x + 2y \leq 2$

$x - 2y \leq 2$

Combining inequalities in two variables 2



$y \geq 2$

$2x + 2y \geq 2$

$x - 2y \geq 1$

Real-life problems

A ferry cannot hold more than 30 tonnes. If it holds x cars weighing 1 tonne each and y lorries weighing 3 tonnes each write down an inequality in x and y.

$$x + 3y \leq 30$$

If 20 cars were already on board how many more lorries could the ferry carry?

Substituting into $x + 3y < 30$ and solving for y,

$$20 + 3y \leq 30$$

subtract 20 from both sides: $3y \leq 10$

divide both sides by 3: $y \leq 3.3$ (to 1 d.p.)

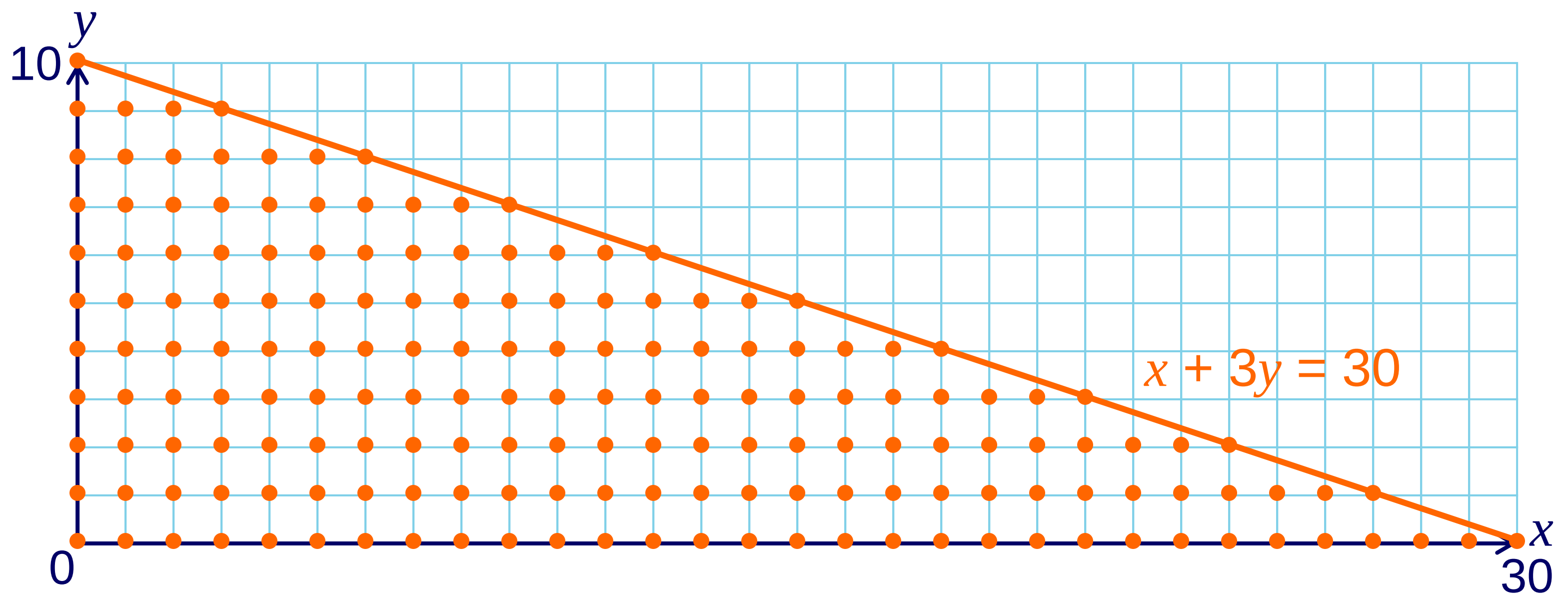
The ferry can hold 3 more lorries.

Real-life problems

Show the possible numbers of cars and lorries that the ferry can carry on a graph.

Start by drawing the x-axis between 0 and 30 and y-axis between 0 and 10.

Next draw the line $x + 3y = 30$



The points on the graph represent the solution set.

Contents

A4 Inequalities

Quadratic inequalities

When inequalities contain terms in x^2 there are usually two solutions. For example,

$$x^2 < 4$$

Remember, $x \times x = x^2$ and $-x \times -x = x^2$

So we can write

$(x)^2 < 4$ and $(-x)^2 < 4$

take the square root: $x < 2$ $-x < 2$

× –1 and reverse: $x > -2$

These solutions can be combined to give $-2 < x < 2$.

Quadratic inequalities

Suppose we have

$$x^2 > 4$$

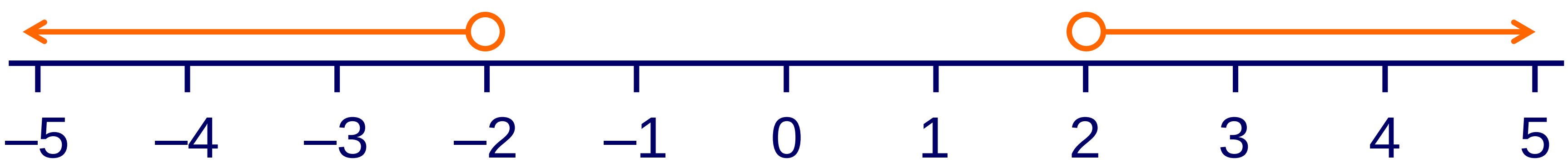
We can write

$(x)^2 > 4$ and $(-x)^2 > 4$

take the square root: $x > 2$ $-x > 2$

× –1 and reverse: $x < -2$

This solution is in two separate parts:



Quadratic inequalities

In more difficult problems there can be terms in both x^2 and x.

$$x^2 + x - 6 \geq 0$$

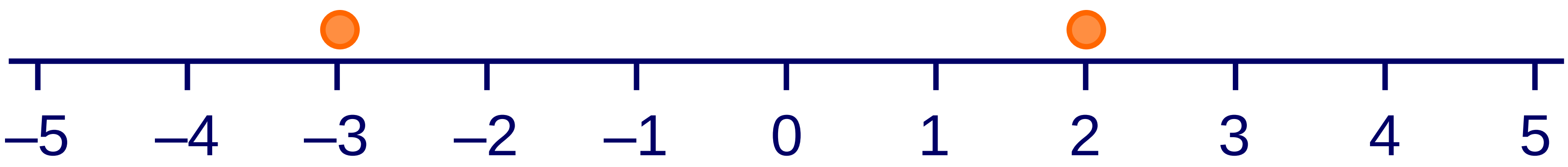
factorize: $$(x + 3)(x - 2) \geq 0$$

The inequality is equal to 0 when:

$x + 3 = 0$ or $x - 2 = 0$

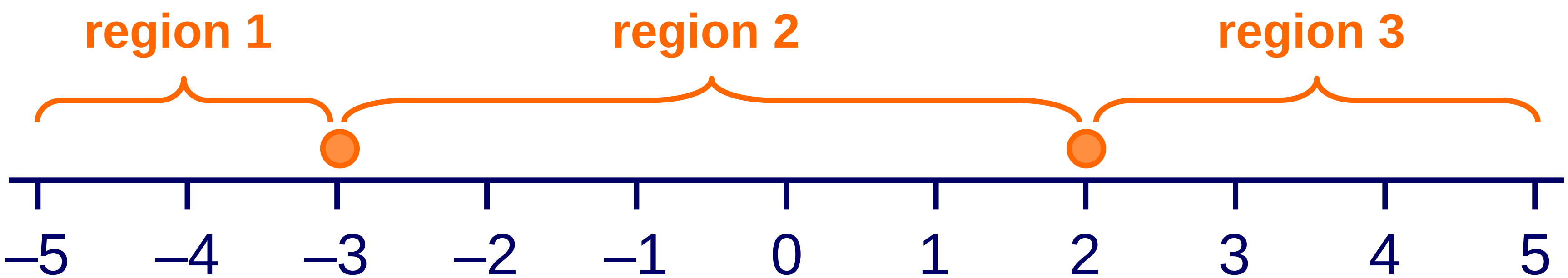
$x = -3$ $x = 2$

These values give the end points of the solution set:



Quadratic inequalities

To find the solution set substitute a value from each of the following three regions



into the original inequality, $x^2 + x - 6 \geq 0$.

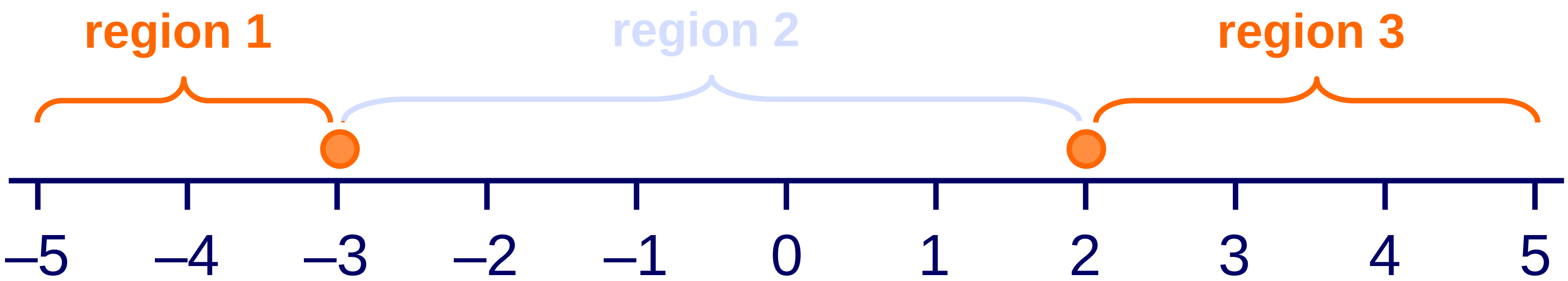
When $x = -4$,

$$-4^2 + -4 - 6 \geq 0$$
$$16 - 4 - 6 \geq 0$$
$$6 \geq 0$$

This is true and so values in **region 1** satisfy the inequality.

Quadratic inequalities

To find the solution set substitute a value from each of the following three regions



into the original inequality, $x^2 + x - 6 \geq 0$.

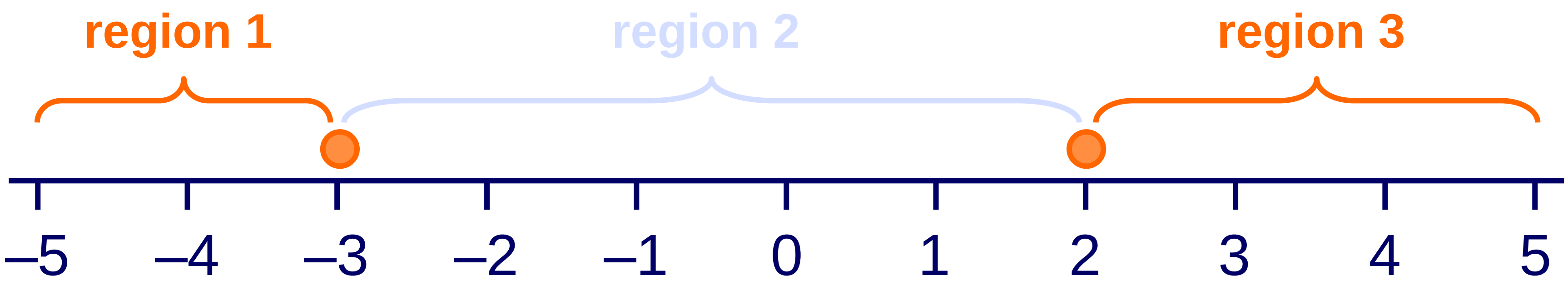
When $x = 0$, $0^2 + 0 - 6 \geq 0$

$-6 \geq 0$

This is not true and so values in **region 2** do not satisfy the inequality.

Quadratic inequalities

To find the solution set substitute a value from each of the following three regions



into the original inequality, $x^2 + x + 6 \geq 0$.

When $x = 3$,

$$3^2 + 3 + 6 \geq 0$$

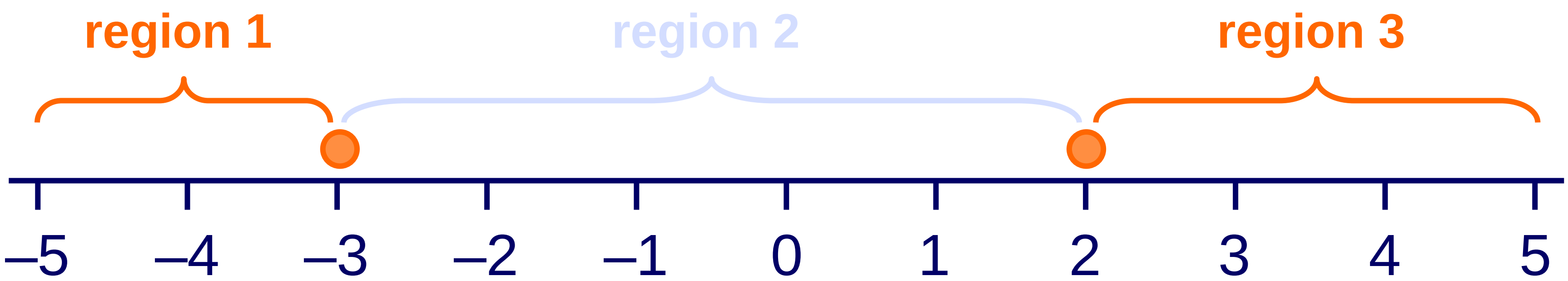
$$9 + 3 + 6 \geq 0$$

$$18 \geq 0$$

This is true and so values in **region 3** satisfy the inequality.

Quadratic inequalities

We have shown that values in **region 1** and **region 3** satisfy the inequality $x^2 + x + 6 \geq 0$.



We can shown the complete solution set as follows: