CIRCLE THEOREMS

Circle Theorems

Lesson Objectives - Students will:

- Apply your knowledge of circle properties to solving problems.
- Identify and apply circle theorems.

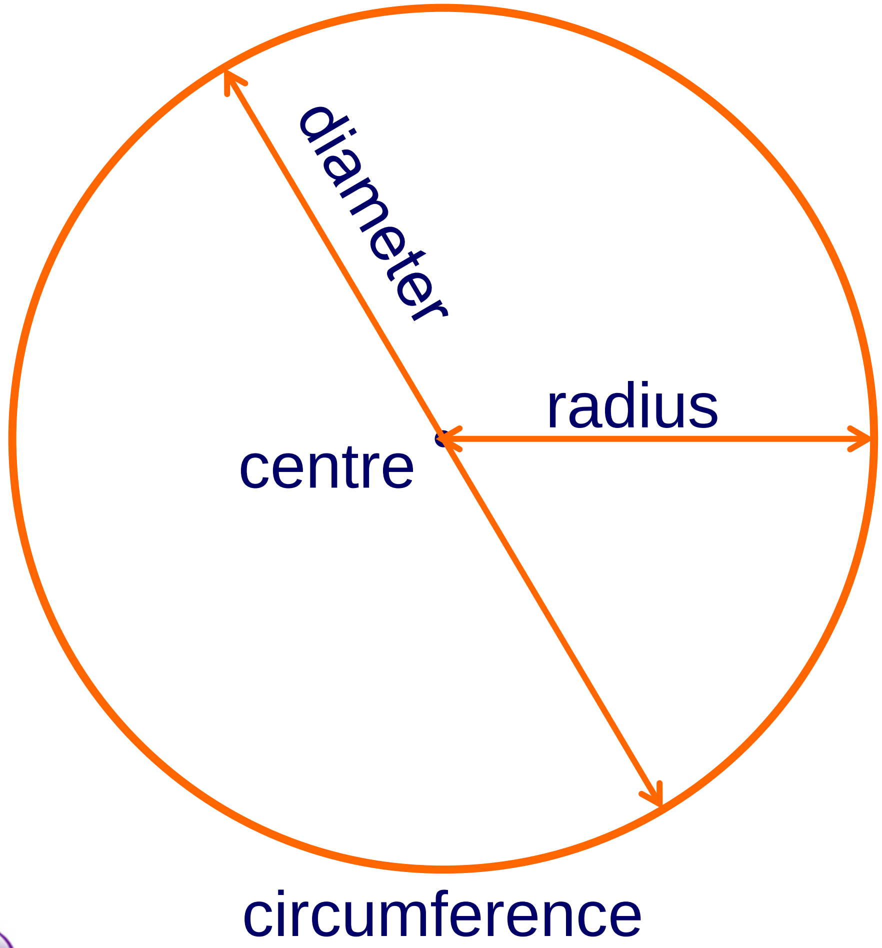
Level 1 LEARN	Complete "Circle Theorem" lesson
Level 2 REVISE	Revise Circle Properties page 408 – 409 Theorem 1 – Subtended angle at centre (p. 411), Theorem 2 – Angle in a semi-circle (p. 412), Theorem 3 – Angle in same segment (p. 415) Theorem 4 – Angles in cyclic quadratic (p. 415) (GCSE Math TEXT – red and blue) Try these: Page 410 #2,3 , Page 413 #1,5,6, Page 417 #1,2,4,10
Level 3 TEST	Circle Theorem TEST Go to http://www.bbc.co.uk/schools/gcsebitesize/maths/shapes/ Scroll to "Circle - Higher", then click "Test"

Contents

S5 Circles

Naming the parts of a circle

A **circle** is a set of points **equidistant** from its **centre**.

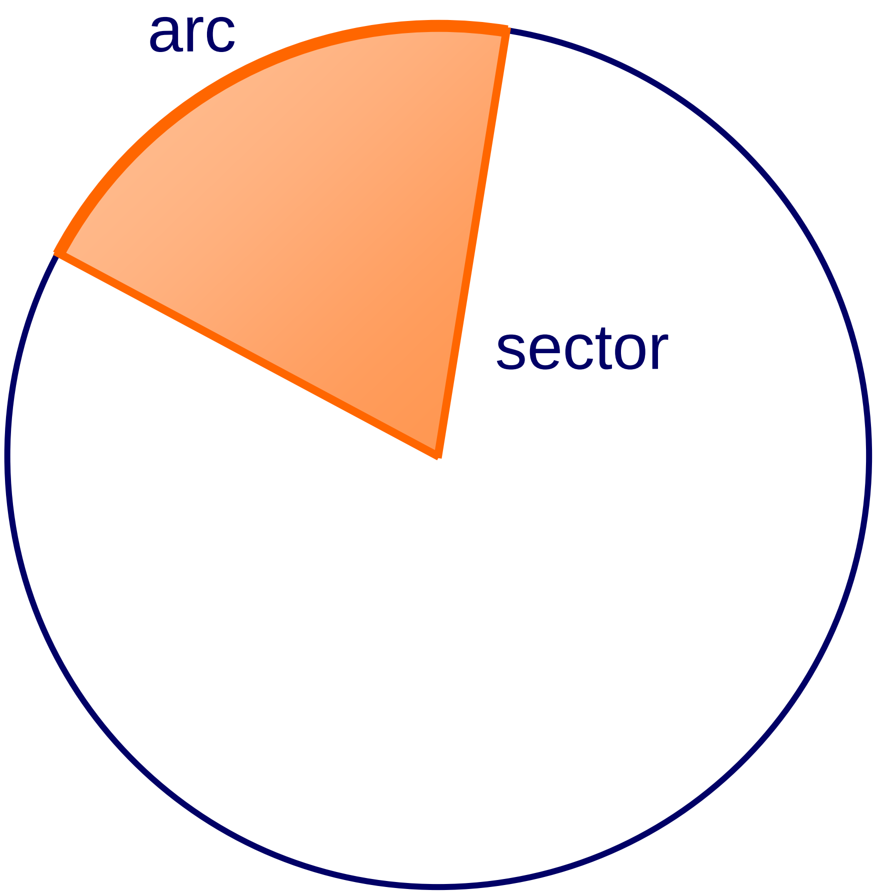


The distance around the outside of a circle is called the **circumference**.

The **radius** is the distance from the centre of the circle to the circumference.

The **diameter** is the distance across the width of the circle through the centre.

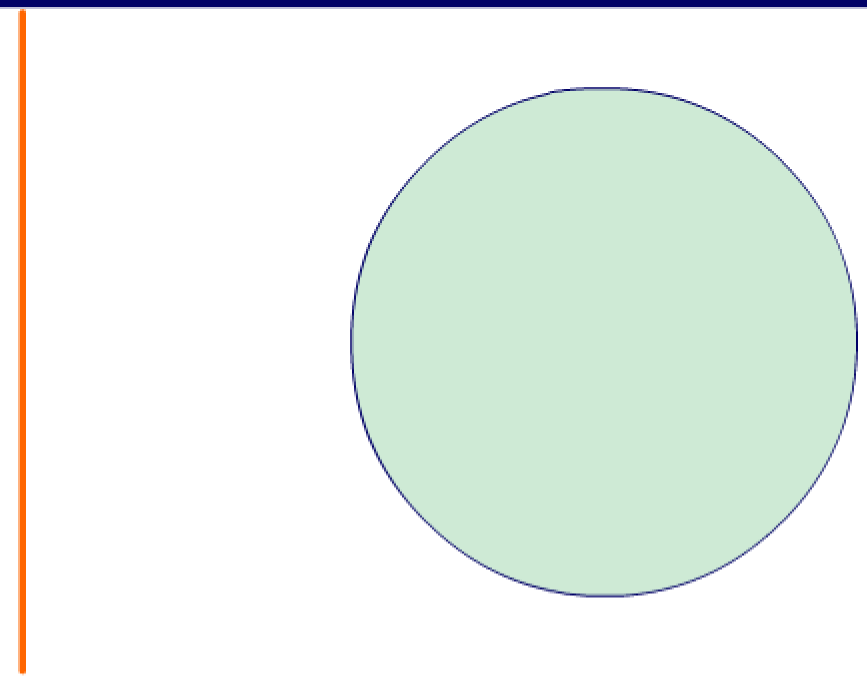
Arcs and sectors



An **arc** is a part of the circumference.

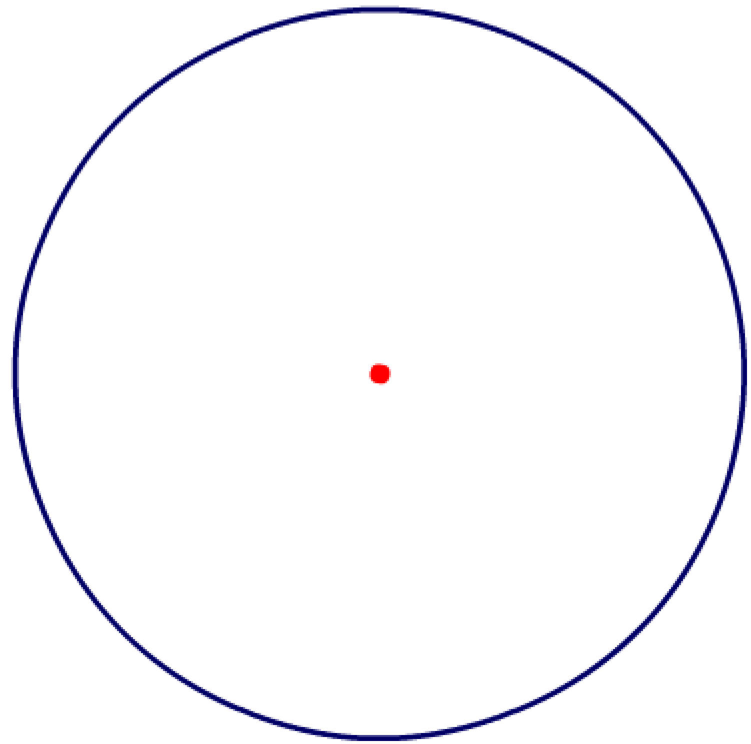
When an arc is bounded by two **radii** a **sector** is formed.

A line moving through a circle



Naming the parts of a circle

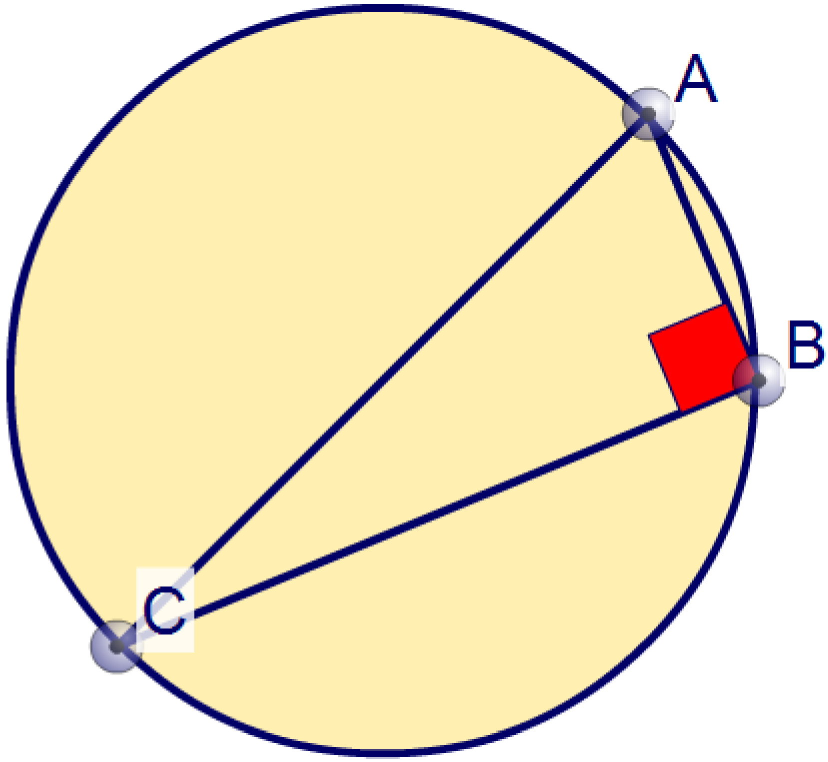
centre
radius
circumference
diameter
arc
sector
segment
semicircle
tangent
chord



Contents

S5 Circles

Right angles in a semicircle

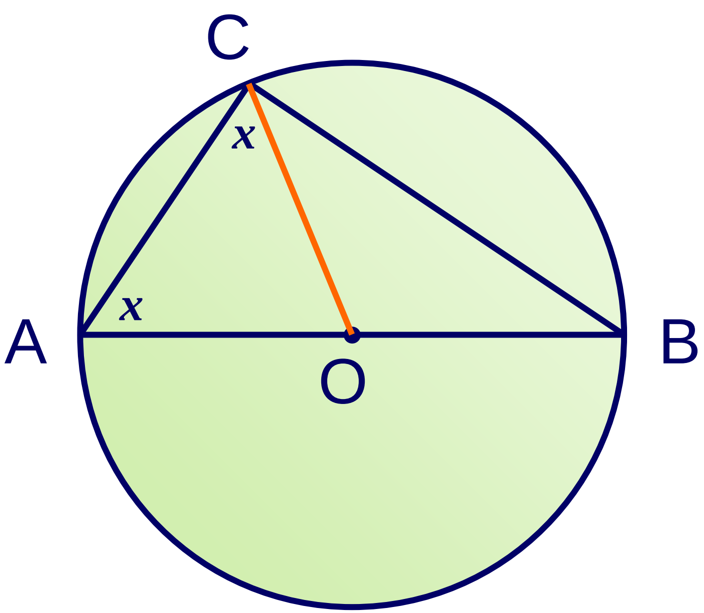


The angle in a semicircle is always 90°

Right angles in a semicircle

We have just seen a demonstration that the angle in a semicircle is always a right angle.

We can **prove** this result as follows:



Draw a line from C to O. This line is a radius of the circle.

In triangle AOC,

OA = OC (both radii)

So, angle OAC = angle OCA

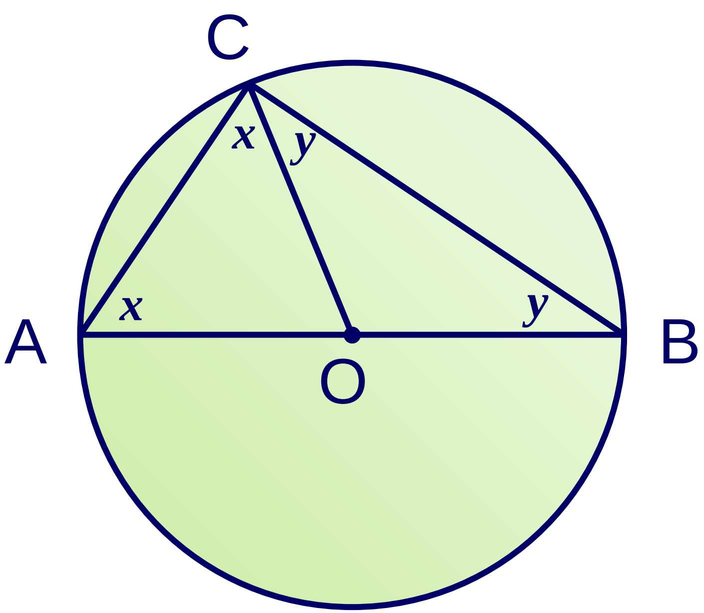
(angles at the base of an isosceles triangle)

Let's call these angles x.

Right angles in a semicircle

We have just seen a demonstration that the angle in a semicircle is always a right angle.

We can **prove** this result as follows:



In triangle BOC,

OB = OC (both radii)

So, angle OBC = angle OCB

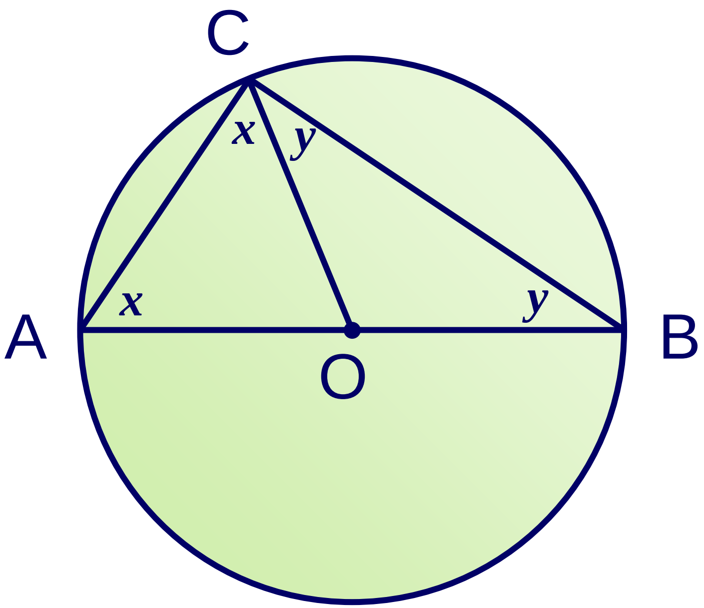
(angles at the base of an isosceles triangle)

Let's call these angles y.

Right angles in a semicircle

We have just seen a demonstration that the angle in a semicircle is always a right angle.

We can **prove** this result as follows:



In triangle ABC,

$x + x + y + y = 180°$ (angles in a triangle)

$2x + 2y = 180°$

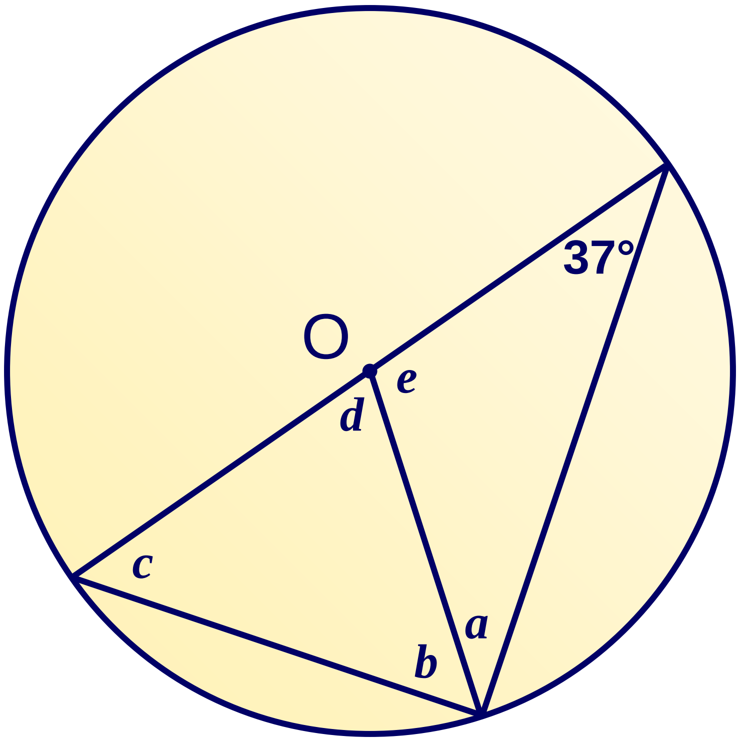
$2(x + y) = 180°$

$x + y = 90°$

Angle ACB $= x + y = 90°$

Calculating the size of unknown angles

Calculate the size of the labeled angles in the following diagram:



a = 37° (angles at the base of an isosceles triangle)

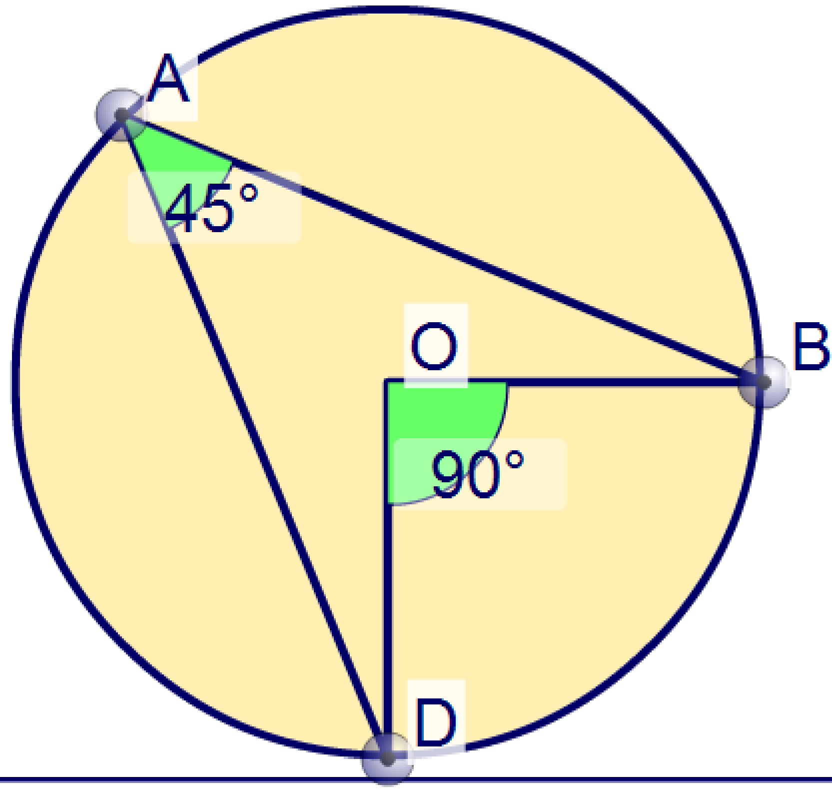
b = 90° – 37°
= 53° (angle in a semi-circle)

c = 53° (angles at the base of an isosceles triangle)

d = 180° – 2 × 53°
= 74° (angles in a triangle)

e = 180° – 74°
= 106° (angles on a line)

The angle at the centre

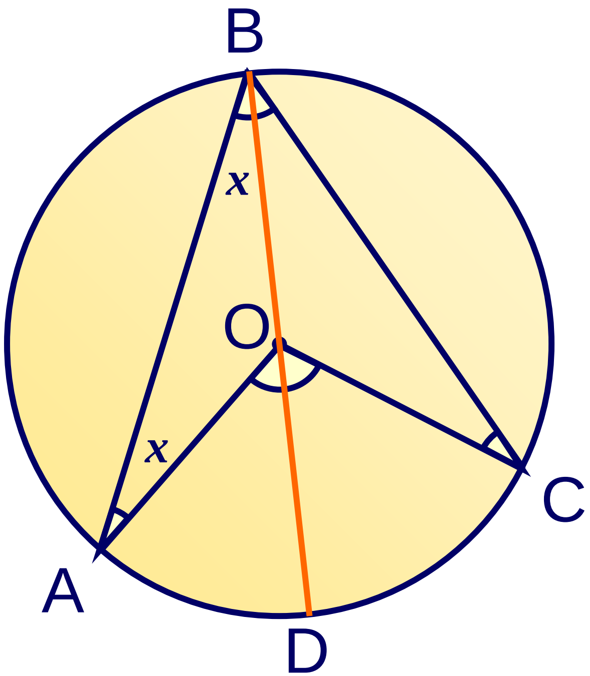


The angle at the centre is twice the angle at the circumference

The angle at the centre

We have just seen a demonstration that the angle at the centre of a circle is twice the angle at the circumference.

We can prove this result as follows:



Draw a line from B, through the centre O, and to the other side D.

In triangle AOB,

OA = OB (both radii)

So, angle OAB = angle OBA

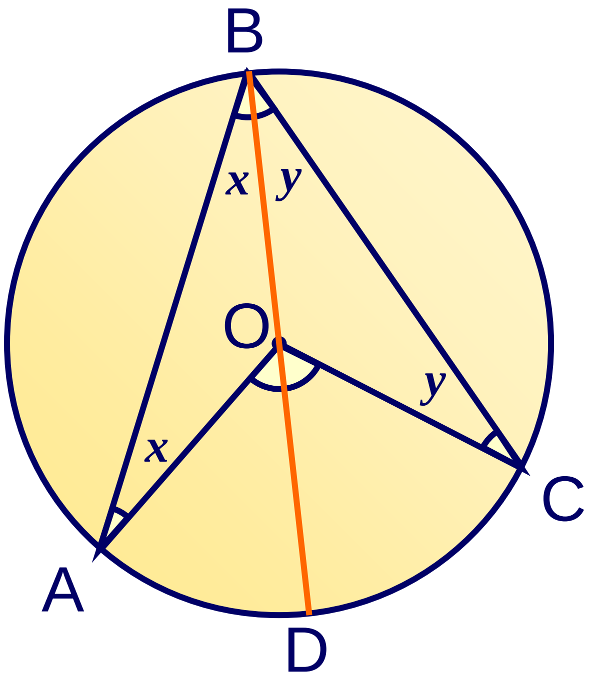
(angles at the base of an isosceles triangle)

Let's call these angles x.

The angle at the centre

We have just seen a demonstration that the angle at the centre of a circle is twice the angle at the circumference.

We can prove this result as follows:



In triangle BOC,

OB = OC (both radii)

So, angle OBC = angle OCB

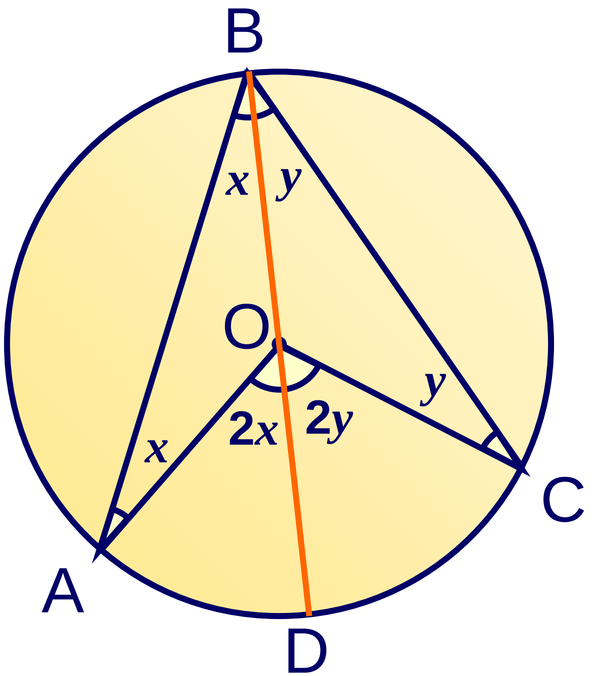
(angles at the base of an isosceles triangle)

Let's call these angles y.

The angle at the centre

We have just seen a demonstration that the angle at the centre of a circle is twice the angle at the circumference.

We can prove this result as follows:



angle AOD = $2x$

and angle COD = $2y$

(the exterior angle in a triangle is equal to the sum of the opposite interior angles)

angle AOC = $2x + 2y$

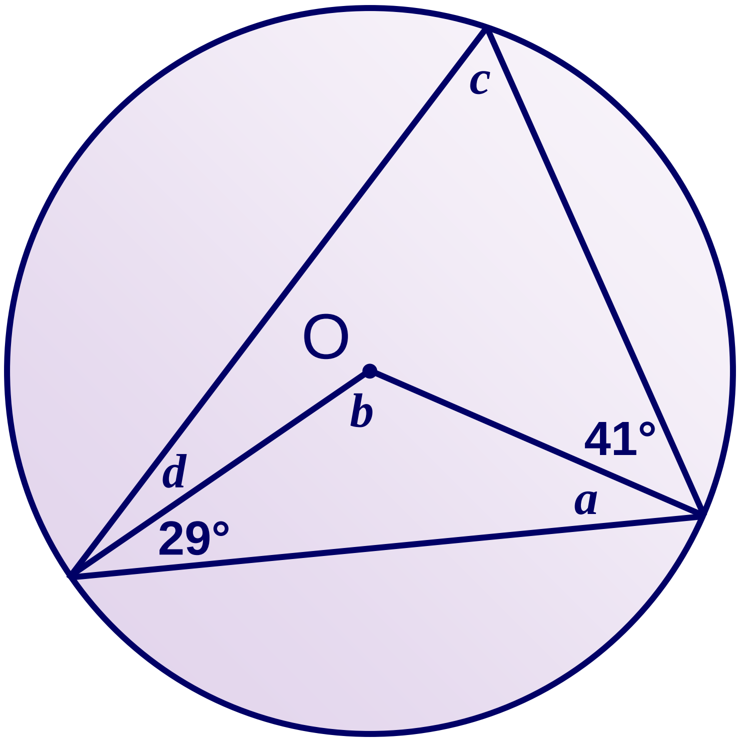
= $2(x + y)$

angle ABC = $x + y$

∴ angle AOC = 2 × angle ABC

Calculating the size of unknown angles

Calculate the size of the labeled angles in the following diagram:



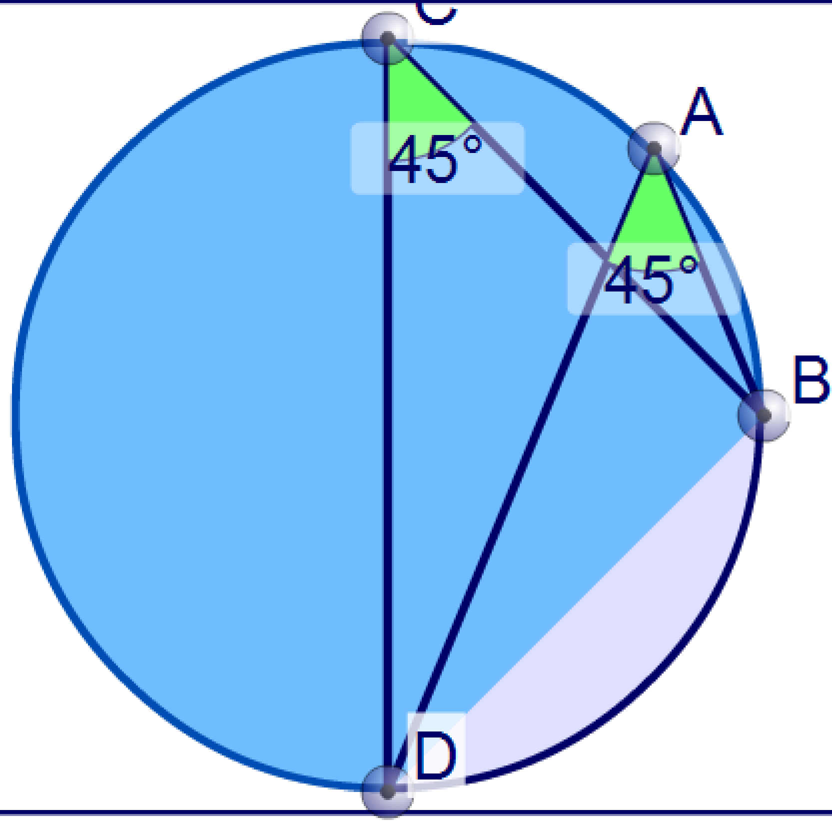
a = 29° (angles at the base of an isosceles triangle)

b = 180° – 2 × 29°
= 122° (angles in a triangle)

c = 122° ÷ 2
= 61° (angle at the centre is twice angle on the circumference)

d = 180° – (29° + 29° + 41° + 61°)
= 20° (angles in a triangle)

Angles in the same segment

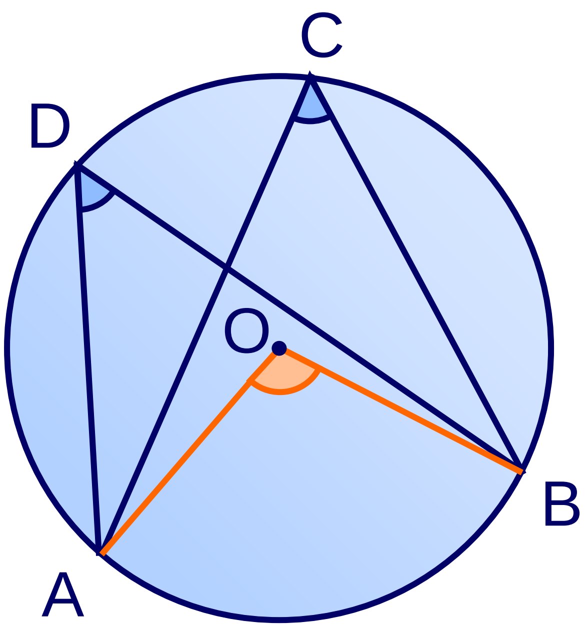


Angles in the same segment are equal.

Angles in the same segment

We have just seen a demonstration that the angles in the same segment are equal.

We can prove this result as follows:



Mark the centre of the circle O and angle AOB.

angle ADB = ½ of angle AOB

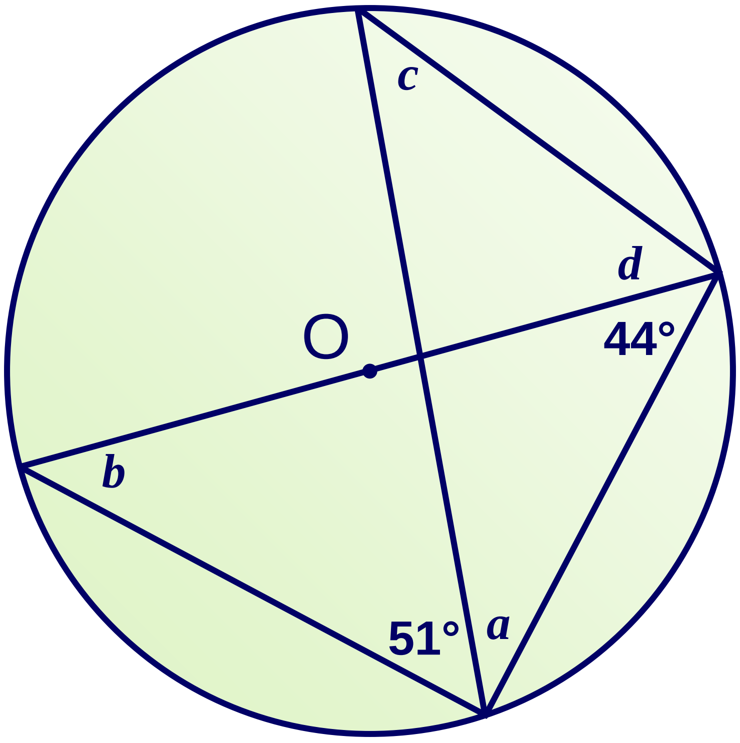
and angle ACB = ½ of angle AOB

(the angle at the centre of a circle is twice the angle at the circumference)

∴ angle ADB = angle ACB

Calculating the size of unknown angles

Calculate the size of the labeled angles in the following diagram:



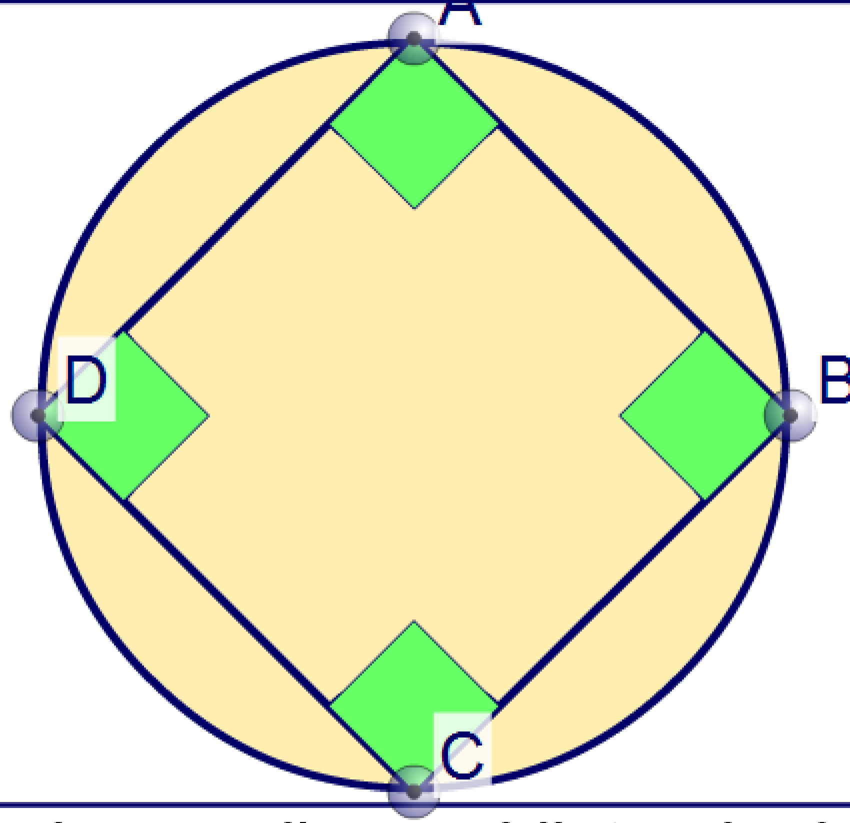
a = 90° – 51°
= 39° (angle in a semi-circle)

b = 180° – (90° + 44°)
= 46° (angles in a triangle)

c = 46° (angles in the same segment)

d = 51° (angles in the same segment)

Angles in a cyclic quadrilateral

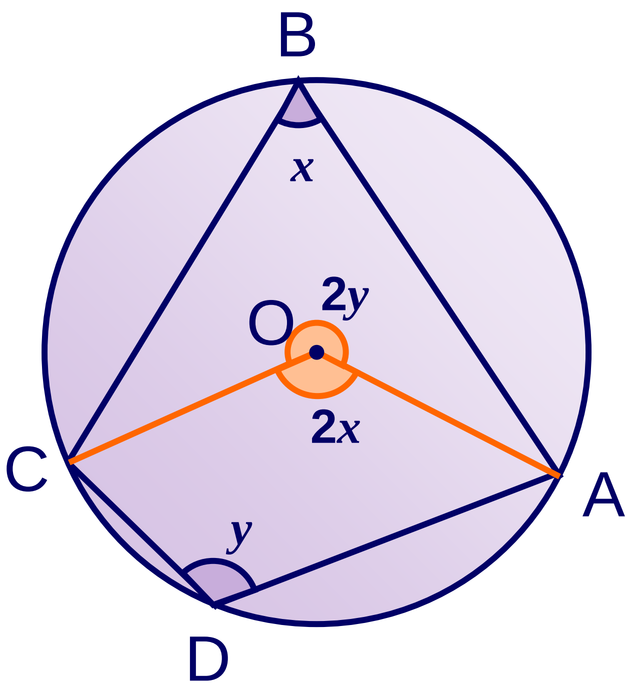


Opposite angles in a cyclic quadrilateral add up to 180°

Angles in a cyclic quadrilateral

We have just seen a demonstration that the opposite angles in a cyclic quadrilateral add up to 180°.

We can prove this result as follows:



Mark the centre of the circle O and label angles ABC and ADC x and y.

The angles at the centre are $2x$ and $2y$.

(the angle at the centre of a circle is twice the angle at the circumference)

$$2x + 2y = 360°$$

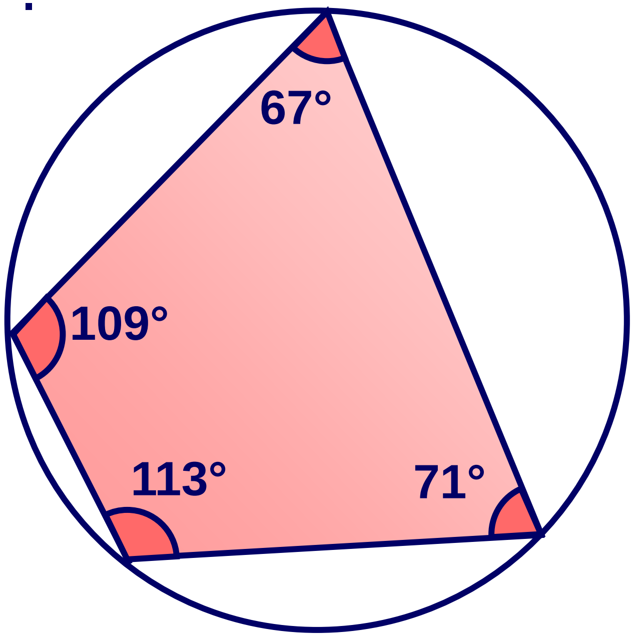
$$2(x + y) = 360°$$

$$x + y = 180°$$

Angles in a cyclic quadrilateral

As a result of this theorem we can conclude that is the opposite angle of a quadrilateral add up to 180° a circle can be drawn through each of its vertices.

For example, the opposite angles in this quadrilateral add up to 180°.

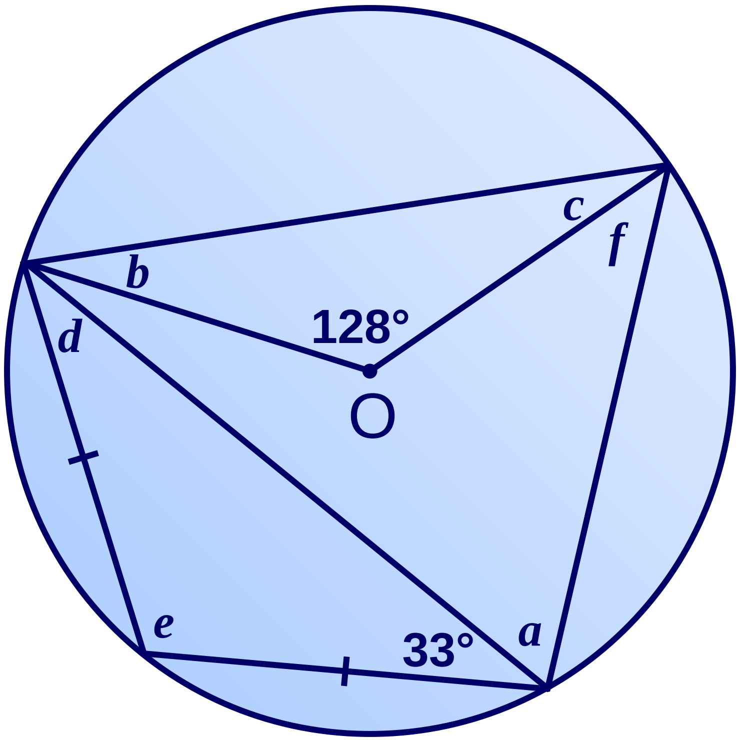


It is a cyclic quadrilateral.

Remember that when two angles add up to 180° they are often called **supplementary angles**.

Calculating the size of unknown angles

Calculate the size of the labeled angles in the following diagram:



a = 64° (angle at the centre)

$b = c$ = (180° – 128°) ÷ 2
= 26° (angles at the base of an isosceles triangle)

d = 33° (angles at the base of an isosceles triangle)

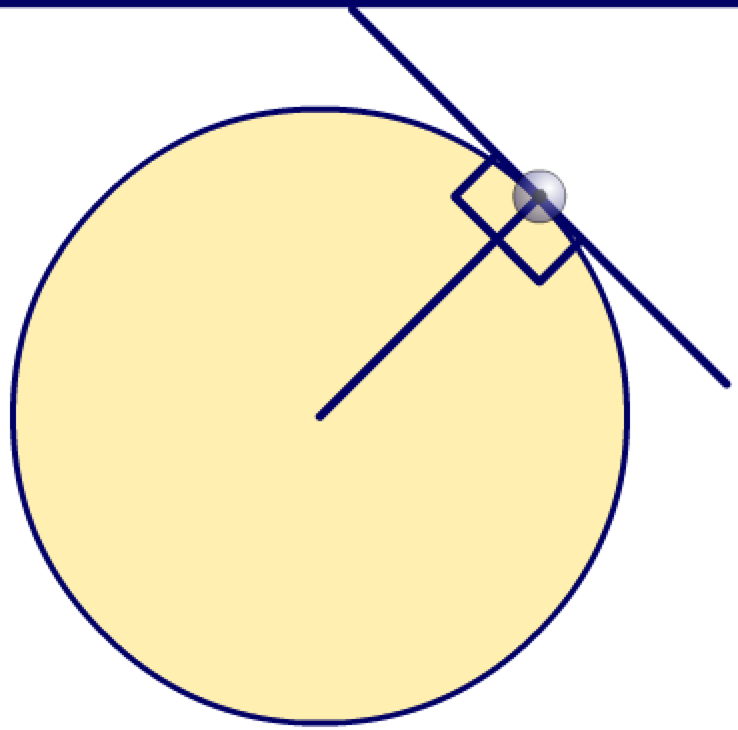
e = 180° – 2 × 33°
= 114° (angles in a triangle)

f = 180° – (e + c)
= 180° – 140°
= 40° (opposite angles in a cyclic quadrilateral)

Contents

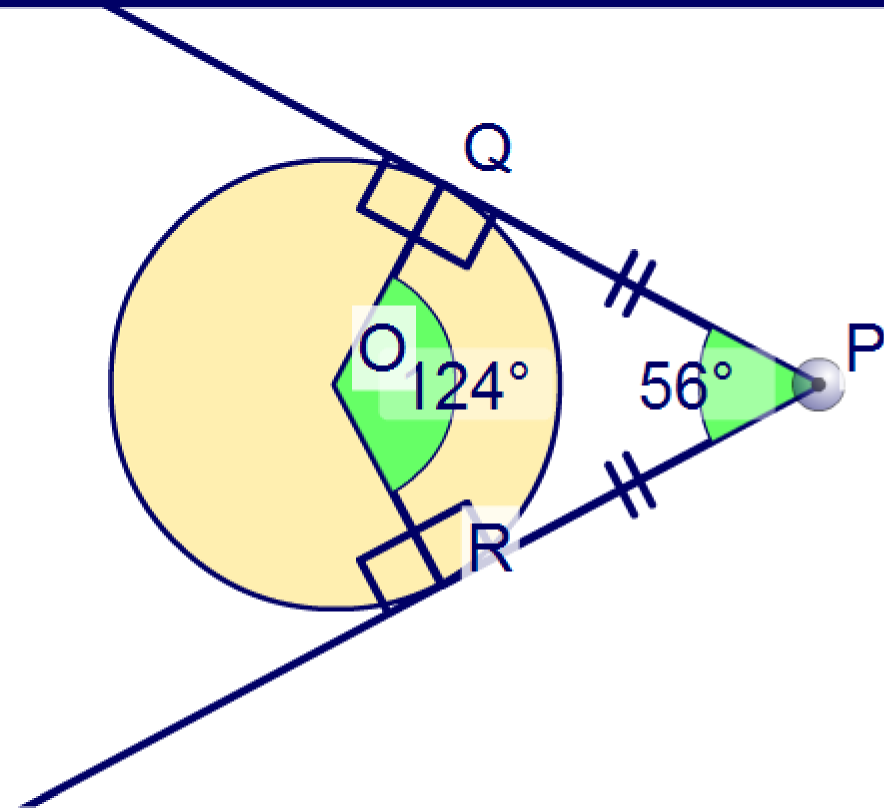
S5 Circles

The tangent and the radius



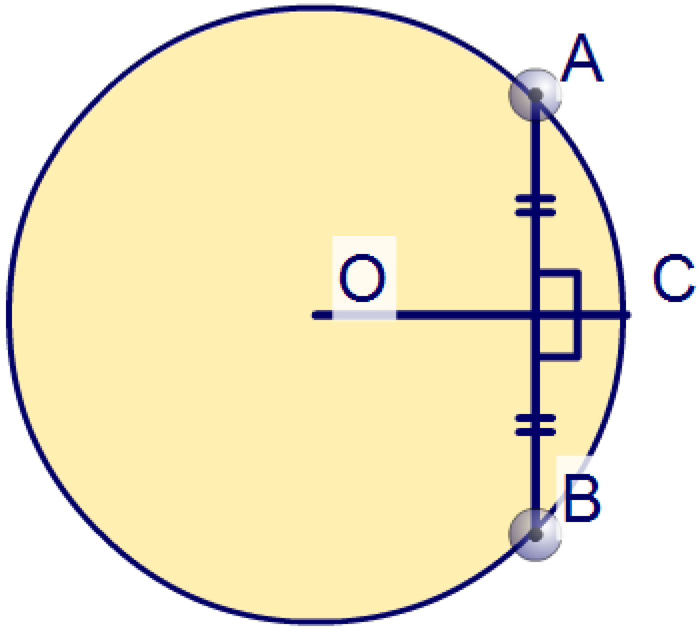
The angle between the tangent and the radius is 90° at the point of contact.

Two tangents from a point



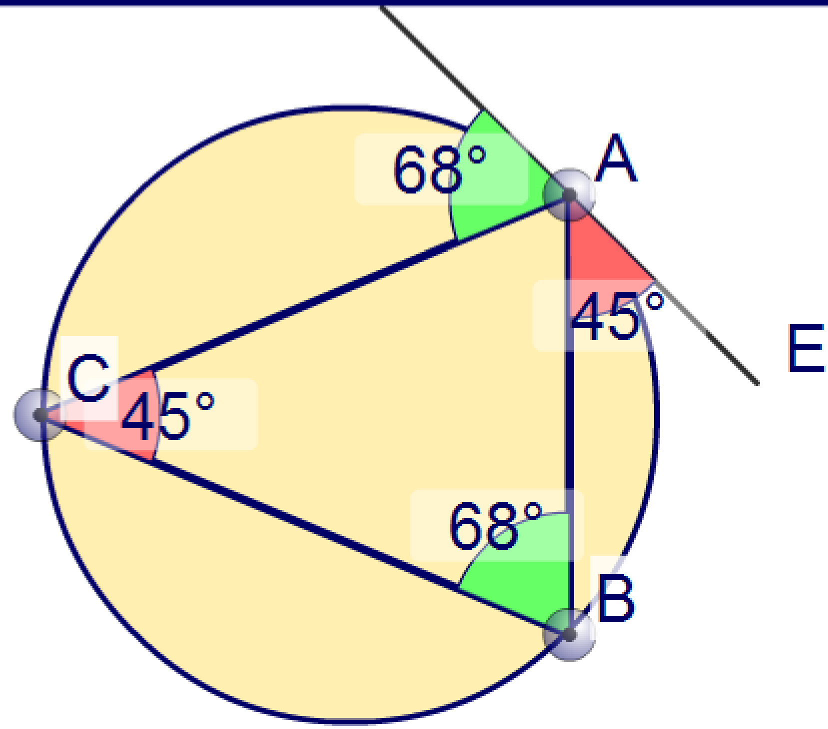
Two tangents from the same point to a circle are of equal length.

The perpendicular from the centre to a chord



The perpendicular from the centre of a circle to a chord bisects the chord.

The alternate segment theorem



The angle between a tangent and a chord is equal to the angle in the alternate segment

Contents

S5 Circles

The value of π

For any circle the circumference is always just over three times bigger than the radius.

The exact number is called π (pi).

We use the symbol π because the number cannot be written exactly.

π = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196 (to 200 decimal places)!

Approximations for the value of π

When we are doing calculations involving the value π we have to use an approximation for the value.

For a rough approximation we can use 3.

Better approximations are 3.14 or $\frac{22}{7}$.

We can also use the π button on a calculator.

Most questions will tell you what approximations to use.

When a calculation has lots of steps we write π as a symbol throughout and evaluate it at the end, if necessary.

The circumference of a circle

For any circle,

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

or,

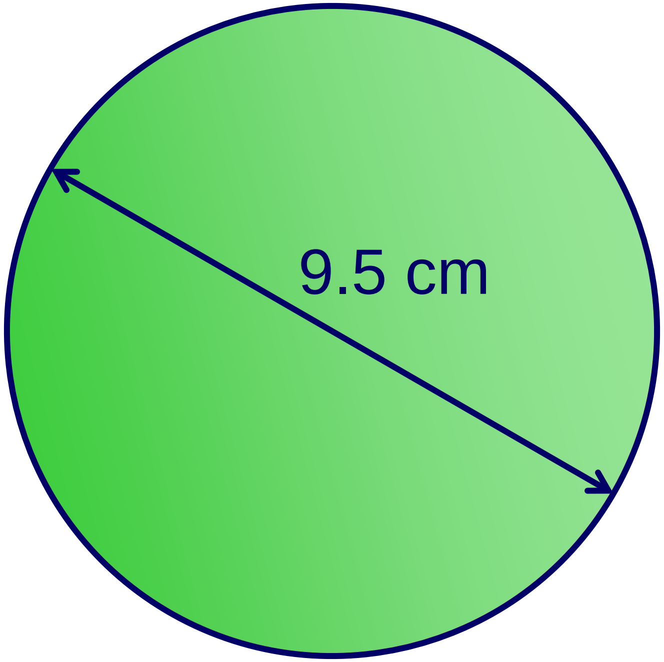
$$\pi = \frac{C}{d}$$

We can rearrange this to make an formula to find the circumference of a circle given its diameter.

$$C = \pi d$$

The circumference of a circle

Use π = 3.14 to find the circumference of this circle.



$C = \pi d$

= 3.14 × 9.5

= **29.83 cm**

Finding the circumference given the radius

The diameter of a circle is two times its radius, or

$$d = 2r$$

We can substitute this into the formula

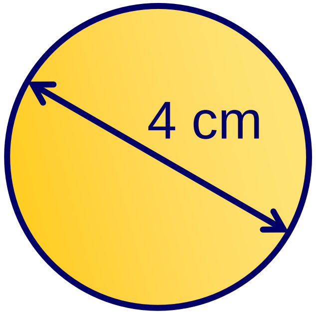
$$C = \pi d$$

to give us a formula to find the circumference of a circle given its radius.

$$C = 2\pi r$$

The circumference of a circle

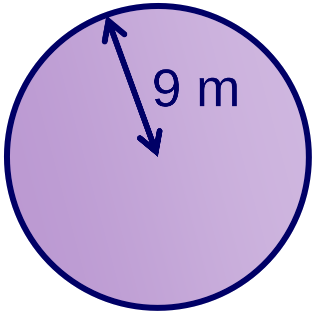
Use $\pi = 3.14$ to find the circumference of the following circles:



$C = \pi d$

$= 3.14 \times 4$

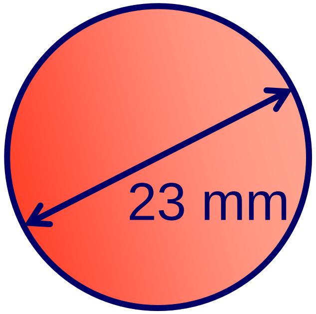
$=$ **12.56 cm**



$C = 2\pi r$

$= 2 \times 3.14 \times 9$

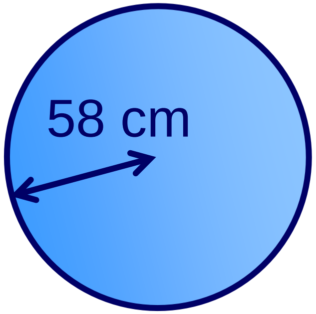
$=$ **56.52 m**



$C = \pi d$

$= 3.14 \times 23$

$=$ **72.22 mm**



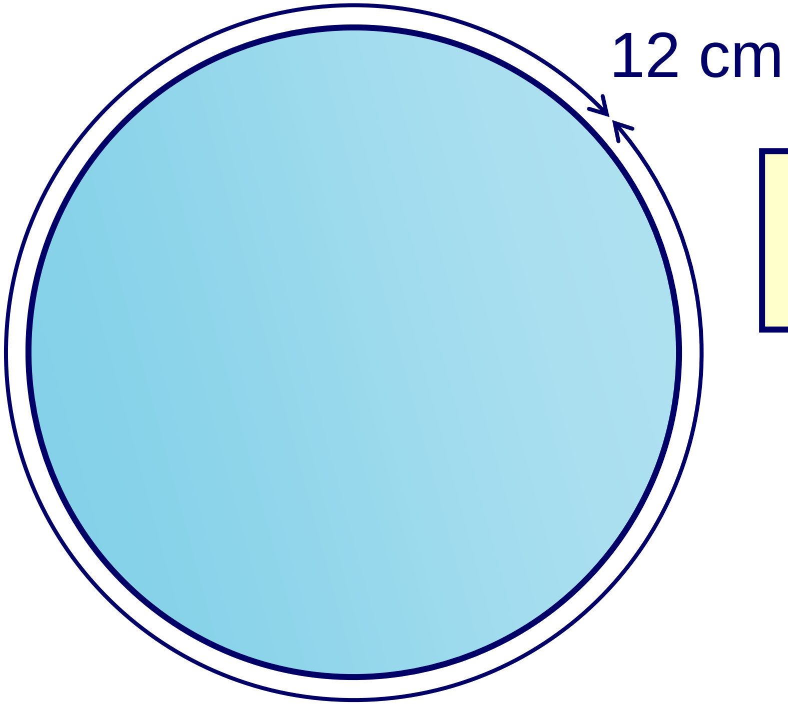
$C = 2\pi r$

$= 2 \times 3.14 \times 58$

$=$ **364.24 cm**

Finding the radius given the circumference

Use $\pi = 3.14$ to find the radius of this circle.



$C = 2\pi r$

How can we rearrange this to make r the subject of the formula?

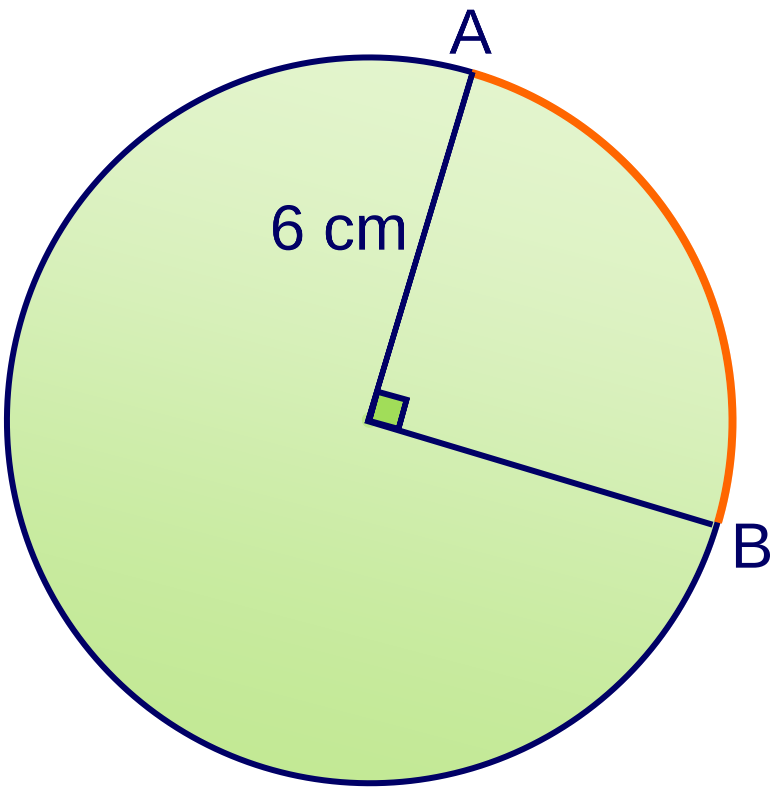
$$r = \frac{C}{2\pi}$$

$$= \frac{12}{2 \times 3.14}$$

= **1.91 cm (to 2 d.p.)**

Finding the length of an arc

What is the length of arc AB?



An arc is a section of the circumference.

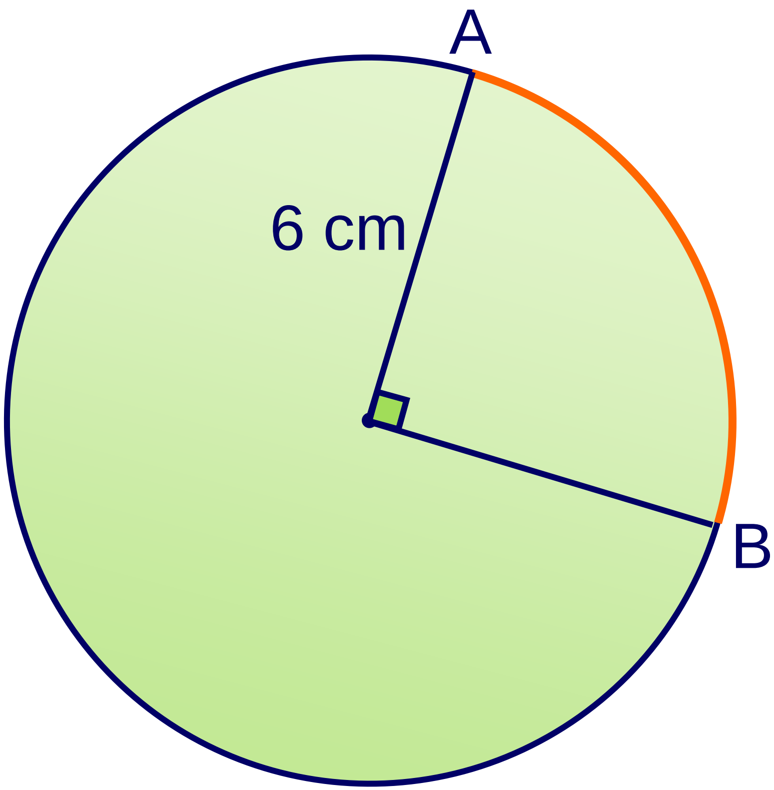
The length of arc AB is a fraction of the length of the circumference.

To work out what fraction of the circumference it is we look at the angle at the centre.

In this example, we have a 90° angle at the centre.

Finding the length of an arc

What is the length of arc AB?



The arc length is $\frac{1}{4}$ of the circumference of the circle.

This is because,

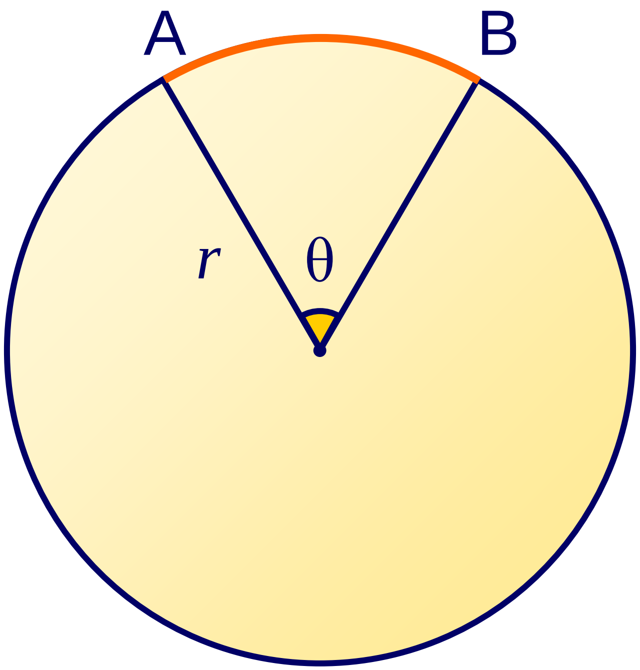
$$\frac{90°}{360°} = \frac{1}{4}$$

So,

$$\text{Length of arc AB} = \frac{1}{4} \times 2\pi r$$
$$= \frac{1}{4} \times 2\pi \times 6$$

Length of arc AB = 9.42 cm (to 2 d.p.)

Finding the length of an arc



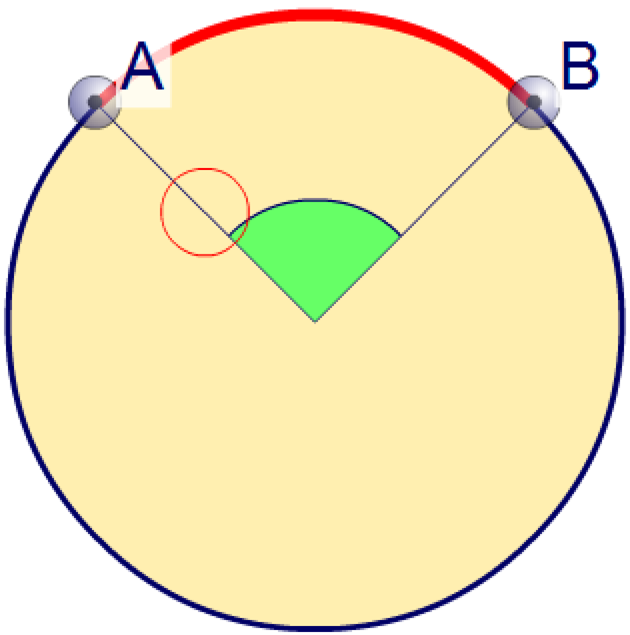
For any circle with radius r and angle at the centre θ,

$$\text{Arc length AB} = \frac{\theta}{360} \times 2\pi r$$

This is the circumference of the circle.

$$\text{Arc length AB} = \frac{2\pi r\theta}{360} = \frac{\pi r\theta}{180}$$

Finding the length of an arc



$$\text{Arc AB} = \frac{2\pi r\theta}{360} = \frac{\pi r\theta}{180}$$

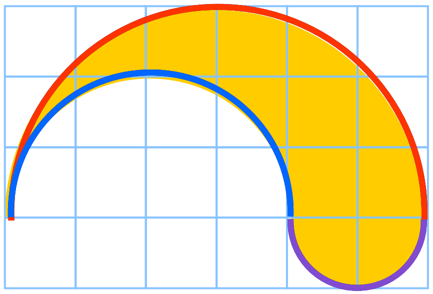
Arc AB =

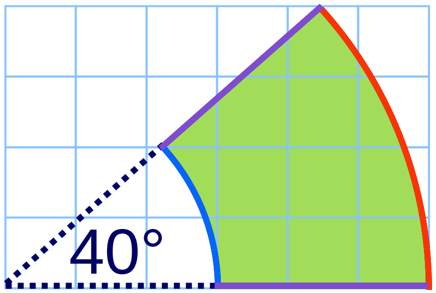
=

5 cm

The perimeter of shapes made from arcs

Find the perimeter of these shapes on a cm square grid:





The perimeter of this shape is made from three semi-circles.

$$\text{Perimeter} = \tfrac{1}{2} \times \pi \times 6 + \tfrac{1}{2} \times \pi \times 4 + \tfrac{1}{2} \times \pi \times 2$$

$$= 6\pi \text{ cm}$$

$$= 18.85 \text{ cm (to 2 d.p.)}$$

$$\frac{40°}{360°} = \frac{1}{9}$$

$$\text{Perimeter} = \tfrac{1}{9} \times \pi \times 12 + \tfrac{1}{9} \times \pi \times 6 + 3 + 3$$

$$= 2\pi + 6$$

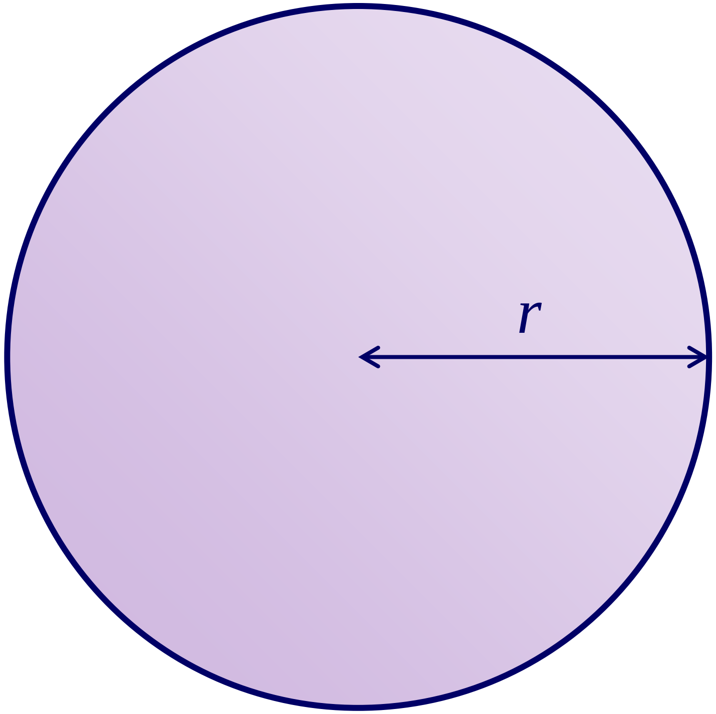
$$= 12.28 \text{ cm (to 2 d.p.)}$$

Contents

S5 Circles

Formula for the area of a circle

We can find the area of a circle using the formula



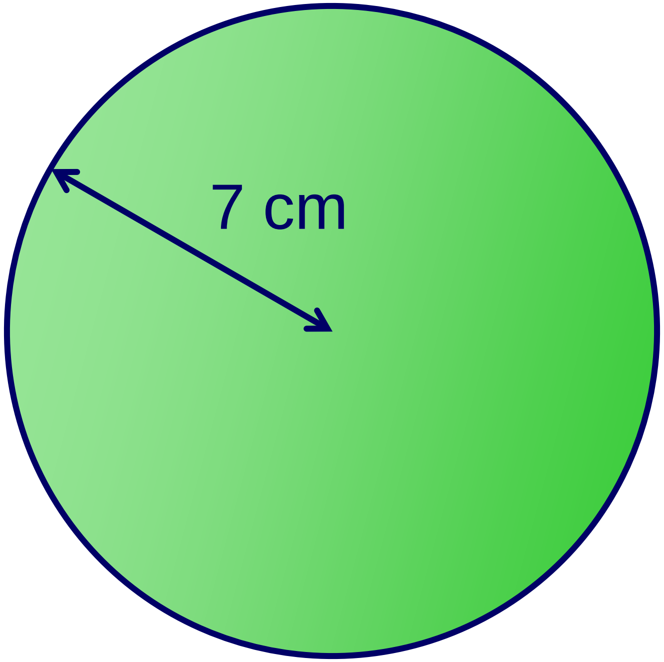
Area of a circle = $\pi \times r \times r$

or

Area of a circle = πr^2

The area of a circle

Use $\pi = 3.14$ to find the area of this circle.



$A = \pi r^2$

$= 3.14 \times 7^2$

$= \mathbf{153.86\ cm^2}$

Finding the area given the diameter

The radius of a circle is half of its radius, or

$$r = \frac{d}{2}$$

We can substitute this into the formula

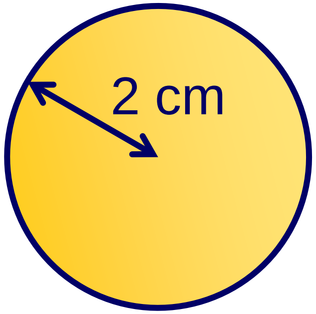
$$A = \pi r^2$$

to give us a formula to find the area of a circle given its diameter.

$$A = \frac{\pi d^2}{4}$$

The area of a circle

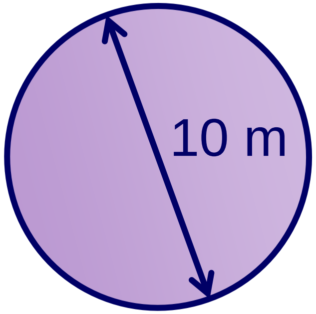
Use $\pi = 3.14$ to find the area of the following circles:



$A = \pi r^2$

$= 3.14 \times 2^2$

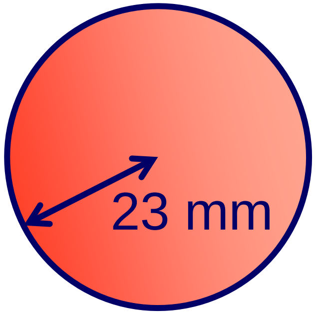
$= $ **12.56 cm²**



$A = \pi r^2$

$= 3.14 \times 5^2$

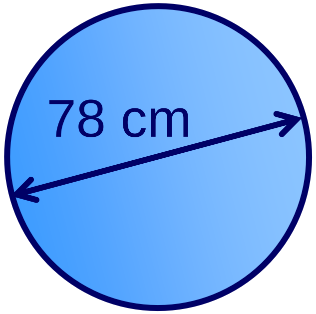
$= $ **78.5 m²**



$A = \pi r^2$

$= 3.14 \times 23^2$

$= $ **1661.06 mm²**



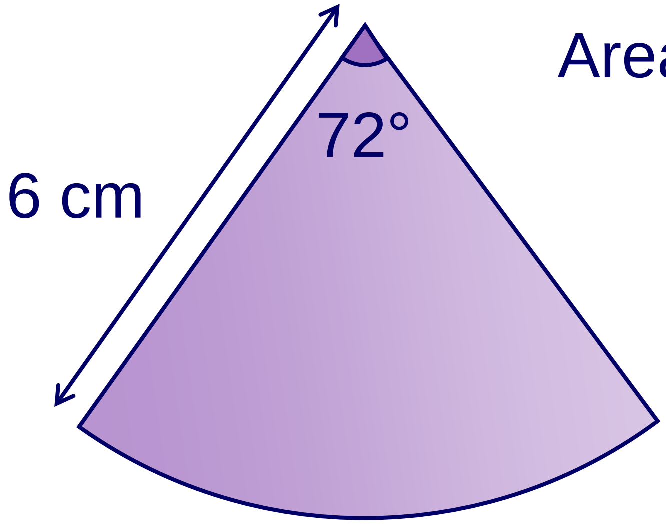
$A = \pi r^2$

$= 3.14 \times 39^2$

$= $ **4775.94 cm²**

Finding the area of a sector

What is the area of this sector?



$$\text{Area of the sector} = \frac{72°}{360°} \times \pi \times 6^2$$

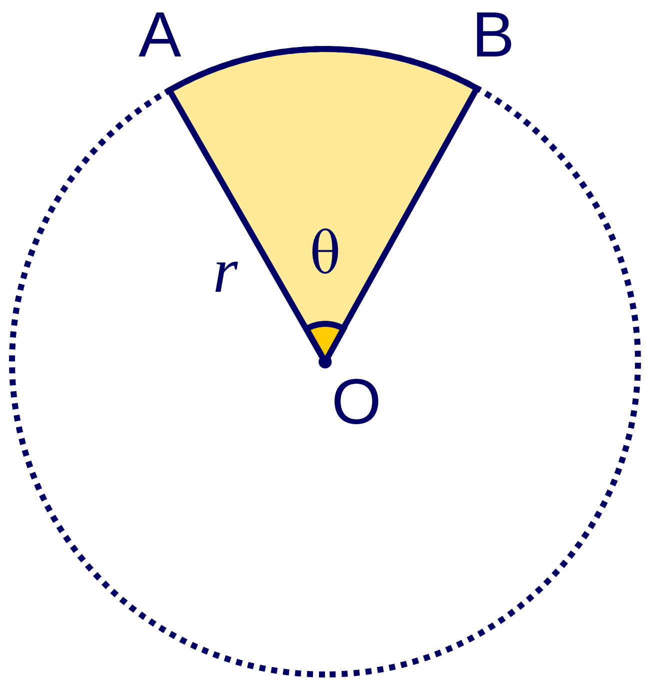
$$= \frac{1}{6} \times \pi \times 6^2$$

$$= \pi \times 6$$

$$= \textbf{18.85 cm}^2 \textbf{ (to 2 d.p.)}$$

We can use this method to find the area of any sector.

Finding the area of a sector



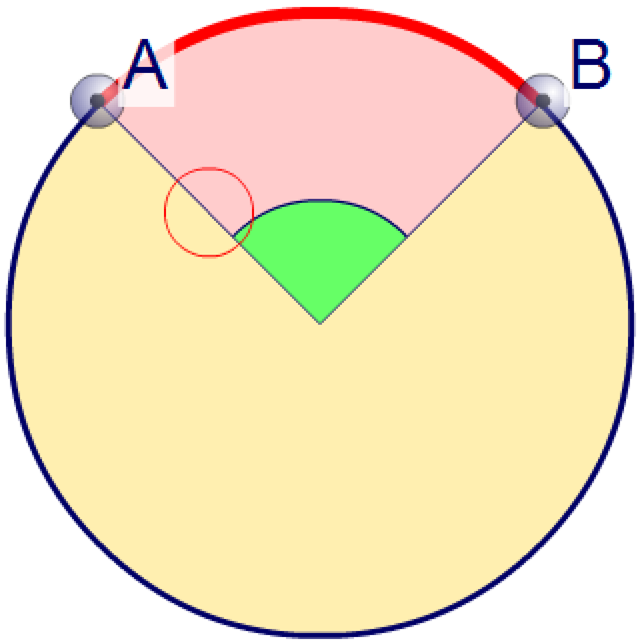
For any circle with radius r and angle at the centre θ,

$$\text{Area of sector AOB} = \frac{\theta}{360} \times \pi r^2$$

This is the area of the circle.

$$\text{Area of sector AOB} = \frac{\pi r^2 \theta}{360}$$

Finding the area of a sector



Area of sector AOB = $\frac{\pi r^2 \theta}{360}$

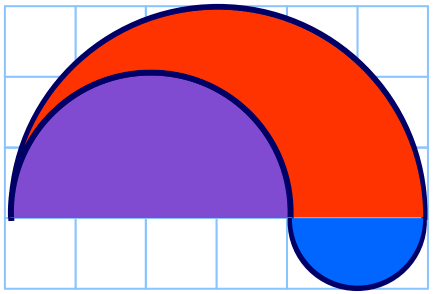
=

=

5 cm

The area of shapes made from sectors

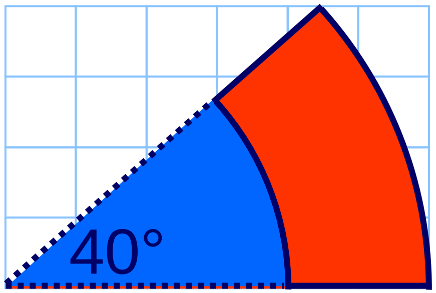
Find the area of these shapes on a cm square grid:



$$\text{Area} = \frac{1}{2} \times \pi \times 3^2 + \frac{1}{2} \times \pi \times 1^2 - \frac{1}{2} \times \pi \times 2^2$$

$= 3\pi$ cm^2

$= 9.42$ cm^2 (to 2 d.p.)



$$\frac{40°}{360°} = \frac{1}{9}$$

$$\text{Area} = \frac{1}{9} \times \pi \times 6^2 - \frac{1}{9} \times \pi \times 4^2$$

$= \frac{1}{9} \times \pi \times 20$ cm^2

$= 6.98$ cm^2 (to 2 d.p.)