- SECONDARY SCHOOL IMPROVEMENT PROGRAMME (SSIP) 2021

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GRADE 12

SUBJECT: MATHEMATICS

TEACHER NOTES TERM 2

TABLE OF CONTENTS

LESSON OVERVIEW

SESSIONS	
Content	**Duration**
Introduction	15 minutes
TOPIC exercises	45 minutes
Feedback	20 minutes
Conclusion Summary	10 minutes
	90 minutes

THE SESSIONS SHOULD BE PROBLEM SOLVING SESSION

LEARNERS SHOULD BE WORKING MORE

SESSION NO 1

TOPIC: ANALYTICAL GEOMETRY

Teaching Tips

Introduction

- The teacher introduces the topic and bring to the learners attention the weighting of the topic in the exam at the end of a year.

Analytical Geometry

- The teacher reminds learners about the formulae .
- Refer to the Notes in the manual below.
- Explain the importance of understanding the question before answering.
- Before answering the question ascertain the understanding of the question .

Learners to work more on exercises and then feedback can be done.

Analytical Geometry - is also referred as Coordinate Geometry
Analytical Geometry is a study of geometric properties using algebra.

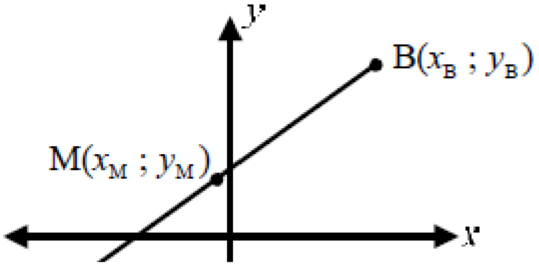
Analytical geometry is a paper 2 topic that carries **40±3 marks** of the total of 150. It involves the following concepts.

- **The distance between two points and the length of a line segment.**

The formula to calculate the length of a line segment between points A and B with $A(x_A ; y_A)$ and $B(x_B ; y_B)$, is:

$$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2 \quad \text{or} \quad AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

- **The mid-point of a line segment.**



Formula

$$M(x_M ; y_M) = M\left(\frac{x_A + x_B}{2} ; \frac{y_A + y_B}{2}\right)$$

The gradient of a line

A formula to calculate the gradient of a line joining two points A and B, with $A(x_A ; y_A)$ and $B(x_B ; y_B)$, is: $m_{AB} = \frac{y_B - y_A}{x_B - x_A}$

- Gradients of perpendicular lines.

For any pair of *perpendicular lines* AB and CD: $m_{AB} \times m_{CD} = -1$

- Gradients of parallel lines.

For any pair of parallel lines AB and CD: $m_{AB} = m_{CD}$

- Relationship of collinear points.

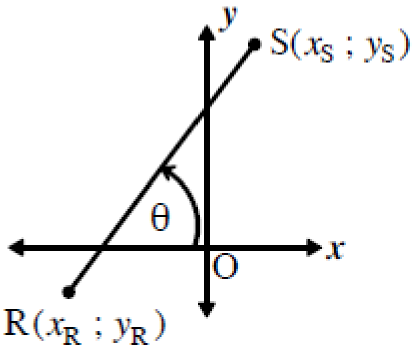
Collinear points are points that lie on the same line.

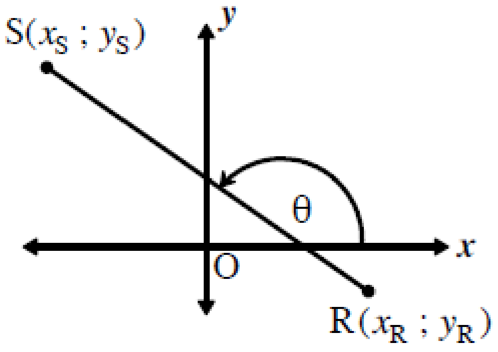
When points A, B and C are collinear: $m_{AB} = m_{AC} = m_{BC}$

In other words: $m_{AB} = m_{AC}$ and $m_{AB} = m_{BC}$ and $m_{AC} = m_{BC}$

- **The angle of inclination of a line and the angle between two lines.**

An angle of inclination is an angle between the line and the positive x - axis





$\therefore \tan\theta = \text{gradient}_{RS}$

$\therefore \tan\theta = \frac{y_S - y_R}{x_S - x_R}$

(a) $y = mx + c$ — m is the gradient and c is the y-intercept (Gradient-intercept form).

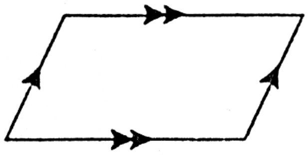
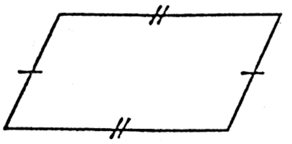
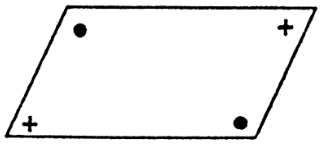
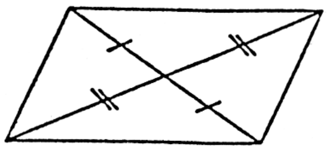
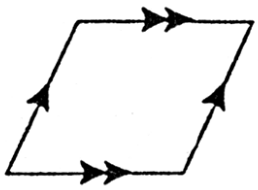
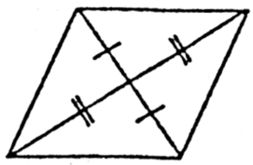
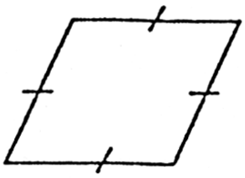
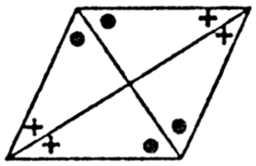
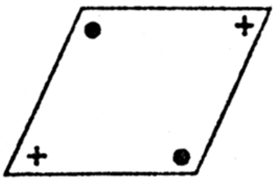
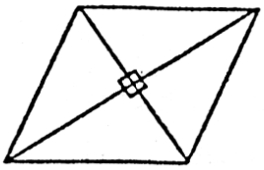
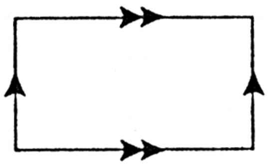
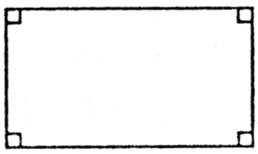
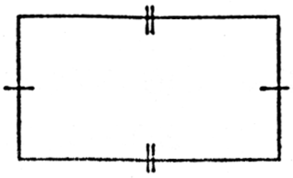
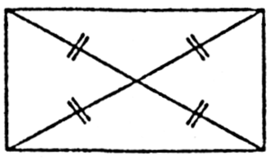
(b) $y - y_1 = m(x - x_1)$ — straight line with gradient m passing throug the point $(x_1 ; y_1)$

(c) Vertical line: $x =$ number — gradient is undefined

(d) Horizontal line: $y =$ number — gradient is zero

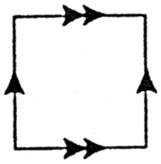
(e) Intercept method: $\frac{x}{a} + \frac{y}{b} = 1$

PROPERTIES OF QUADRILATERALS

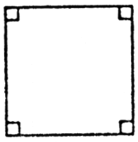
PROPERTIES OF THE SPECIAL QUADRIALTERALS			
1. THE PARALLELOGRAM			
Opposite sides parallel		Opposite sides equal in length	
Opposite angles equal in size		Diagonals bisect each other	
2. THE RHOMBUS			
Opposite sides parallel		Diagonals bisect each other	
All sides equal in length		Diagonals bisect corner angles	
Opposite angles equal in size		Diagonals cross at right angles	
3. THE RECTANGLE			
Opposite sides parallel		All angles equal in size (90°)	
Opposite sides equal in length		Diagonals equal in length and bisect each other	

4. THE SQUARE

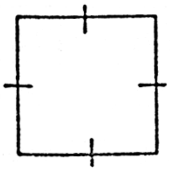
Opposite sides parallel



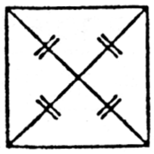
All angles equal in size (90°)



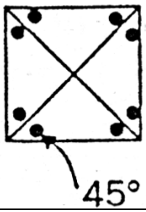
All sides equal in length



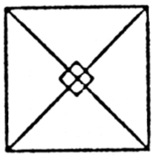
Diagonals equal in length and bisect each other



Diagonals bisect corner angles

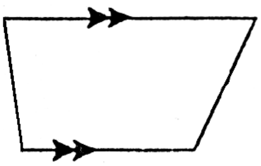


Diagonals cross at right angles



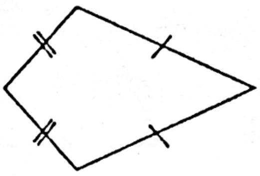
5. THE TRAPEZIUM

One pair of parallel sides

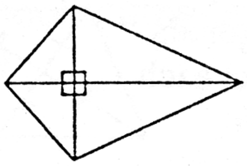


6. THE KITE

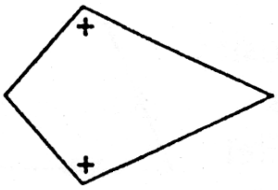
Adjacent sides equal in length



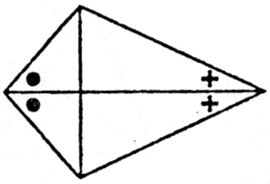
Diagonals cross at right angles



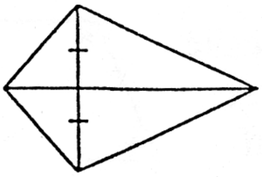
One pair of opposite angles equal in size



Only one pair of opposite angles is bisected



Only one diagonal is bisected



How to solve problems ?

STEP 1

- **Analyse** the diagram

STEP 2

- Analyse the diagram by paying attention to **key words**
- Look for information the diagram which can be **helpful and useful**
- **Use colours** to mark off equal angles /sides
- Look for **implied information**

STEP 3

- Brainstorm and develop a **rough solution**
- **Link** information you have acquired

STEP 4

- Rewrite a **formal calculations**
-

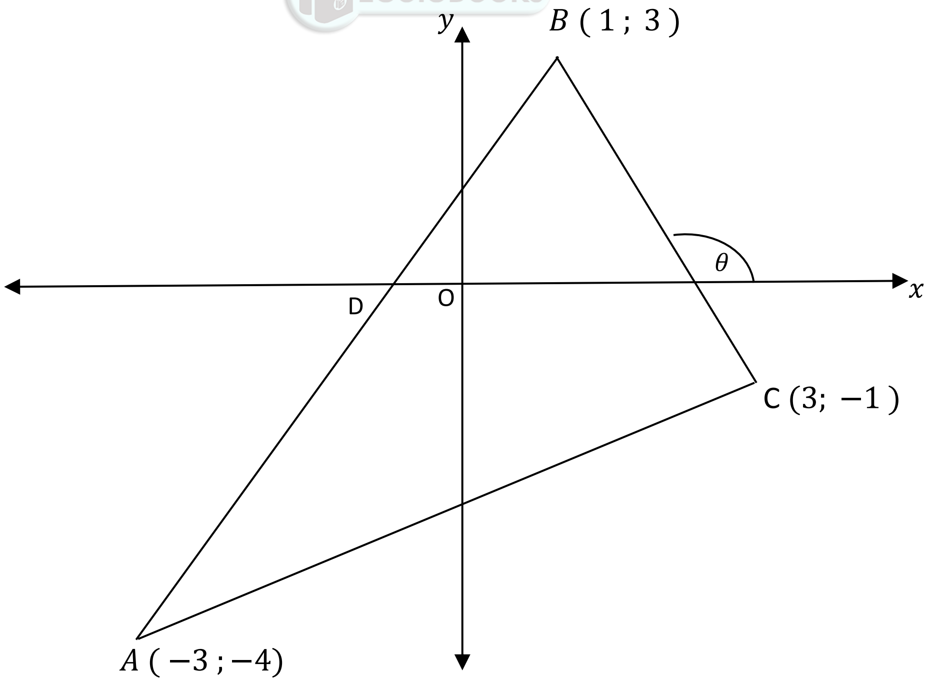
Examples of **KEY** words

- Parallel lines
- Triangle information
- Centre
- Diameter
- Cyclic quad
- Tangents
- Chords etc.

WORKED EXAMPLES:

<u>QUESTION 1</u>

The diagram shows ΔABC with vertices $A(-3\,;-4)$, $B(1\,;3)$ and $C(3\,;-1)$



1.1 Show that the midpoint of AC lies on the y-axis.

1.2 Calculate the magnitude of θ, as shown in the diagram.

1.3 Calculate the co-ordinates of D, the x - intercept of the straight line through A and B

SOLUTIONS

1.1 Midpoint of AC

$$M\left(\frac{x_B + x_A}{2};\frac{y_B + y_A}{2}\right)$$

$$M\left(\frac{-3+3}{2};\frac{-4-1}{2}\right)$$

$$M\left(0;-\frac{5}{2}\right)$$

1.2

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m_{BC} = \frac{3-(-1)}{1-3}$$

$$m_{BC} = -2$$

Guide the learners to make correct substitutions and also to use brackets when substituting negative values.

But $\tan\theta = m$

$$\tan\theta = -2$$

$$Ref\angle = \tan^{-1}(2)$$

$$= 63{,}43°$$

$$\therefore \theta = 180° - 63{,}43°$$

$$\therefore \theta = 116{,}57°$$

1.3

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m_{AB} = \frac{3-(-4)}{1-(-3)}$$

$$m_{AB} = \frac{7}{4}$$

At D; $y = 0$

$$m_{DB} = \frac{0-3}{x-1}$$

$$m_{DB} = \frac{-3}{x-1}$$

Note that A ; D and B are collinear points

$$m_{AB} = m_{DB}$$

$$\frac{7}{4} = \frac{-3}{x-1}$$

$$7(x-1) = -12$$

$$x = -\frac{5}{7}$$

$$\therefore D\left(-\frac{5}{7}; 0\right)$$

OR

$$m_{AB} = \frac{3-(-4)}{1-(-3)}$$

$$m_{AB} = \frac{7}{4}$$

Using $y - y_1 = m(x - x_1)$

At D$(x; 0)$ and B$(1; 3)$

$$0 - 3 = \frac{7}{4}(x-1)$$

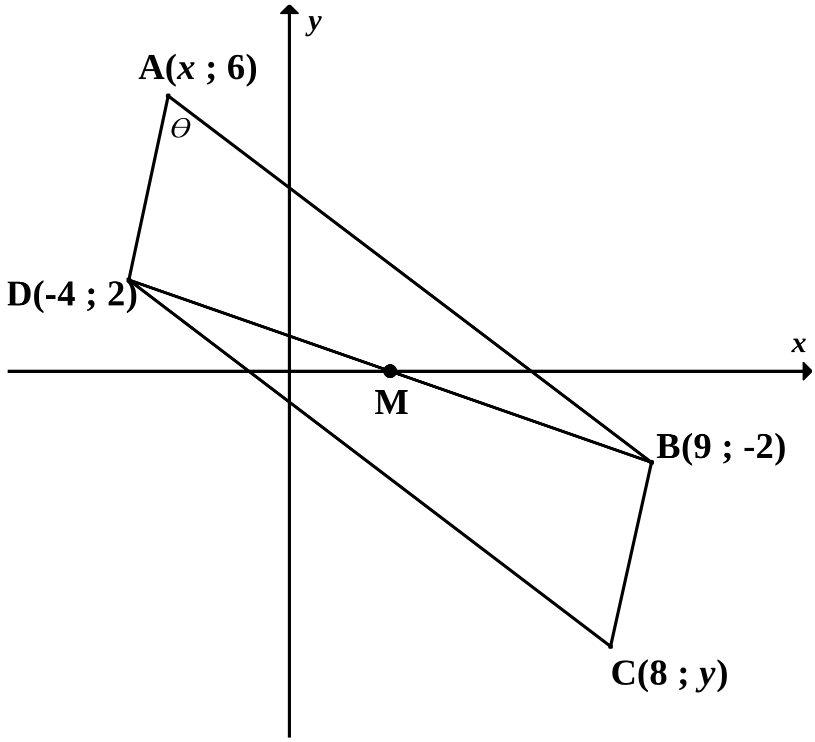
$$x = -\frac{5}{7}$$

$$\therefore D\left(-\frac{5}{7}; 0\right)$$

ACTIVITY 1:

QUESTION 1

In the diagram below ABCD is a parallelogram.



1.1	Calculate the coordinates of M, the midpoint of DB.	(2)
1.2	Hence, or otherwise, calculate the values of *x* and *y*.	(3)
1.3	Calculate the gradient of BC.	(2)
1.4	Calculate the equation of the line AB in the form $y = \ldots$	(4)
1.5	Determine the value of θ, the angle between AD and AB.	(4)
1.6	Prove that Δ DBC is not a right-angled triangle	(3)

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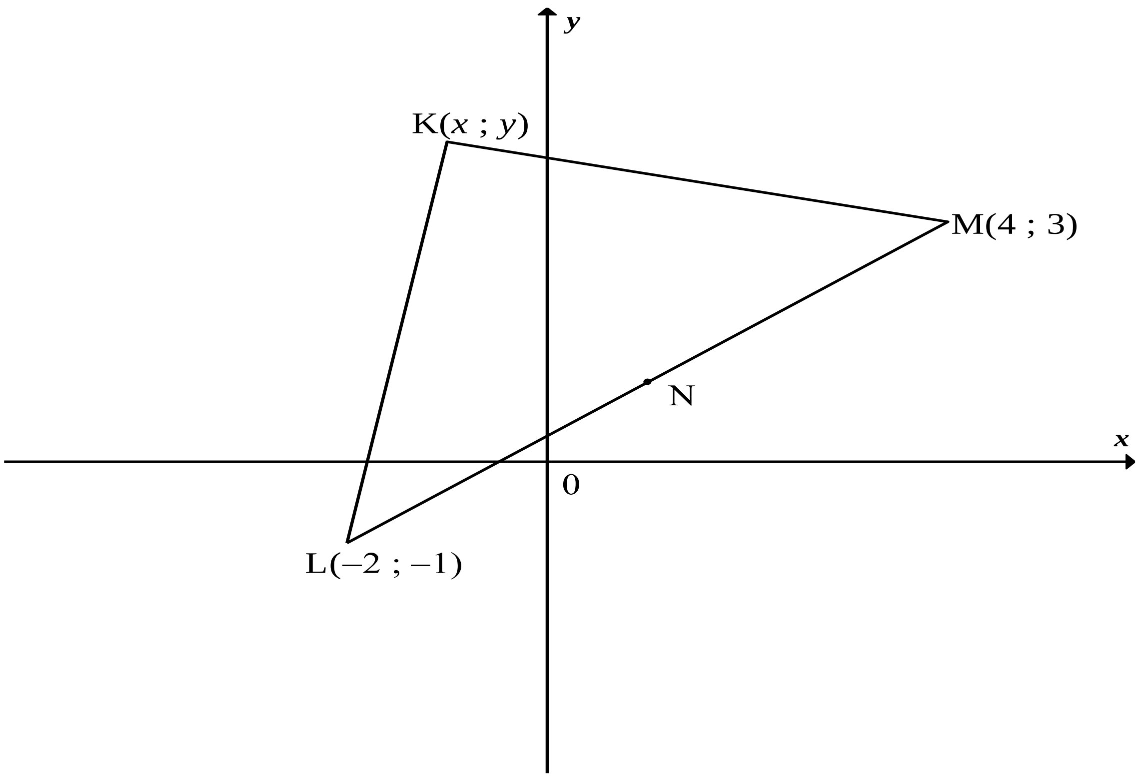
QUESTION 2

In the figure below, $K(x;\ y), L(-2;\ -1)$ and $M(4;\ 3)$ are the vertices of triangle KLM. The equation of the side KL is $y - 5x - 9\ =\ 0$ and that of KM is $5y\ +\ x - 19\ =\ 0$. N is the midpoint of LM.



2.1 Calculate the coordinates of N. (2)

2.2 Show that the coordinates of K are $(-1\ ;\ 4)$. (4)

2.3 Determine the equation of the line through K and N in the form *y* = m*x* + c. (3)

2.4 Determine the gradient of the line LM. Hence, prove that KN is the perpendicular bisector of LM . (3)

2.5 If L, M and the point J(7; *a*) are collinear, calculate the value of *a*. (3)

2.6 Determine the size of the angle of inclination between KL and the positive $x - axis$. (2)

[17]

QUESTION 3

3.1 The line AB has equation $3x-4y+5=0$.

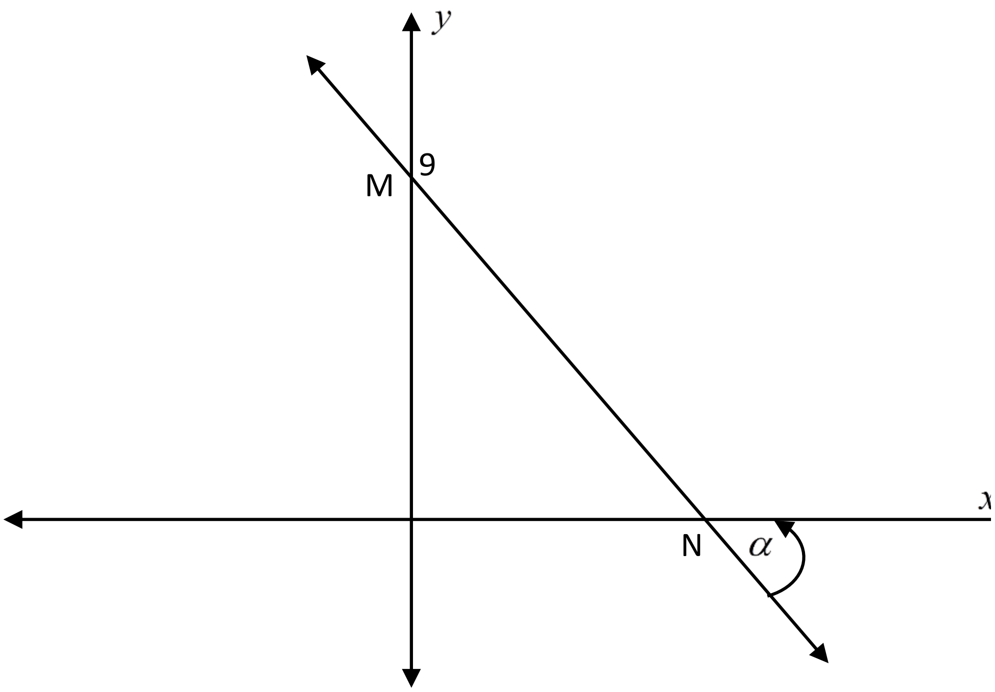
3.1.1 The point with coordinates $(p;p+2)$ lies on the line AB.
Calculate the value of the constant p. (3)

3.1.2 Write down the gradient of line AB. (1)

3.1.3 The point A has coordinates $(1;2)$. The point $C(-5;k)$ is such that AC is perpendicular to AB. Calculate the value of k. (3)

3.2 In the diagram below, MN is a straight line with equation $ax+by+c=0$.

M$(0\,;\,9)$ is the *y*-intercept of the straight line. It is further given that $\tan\alpha=3$.



Calculate the values of *a*, *b* and *c*. (6)

[13]

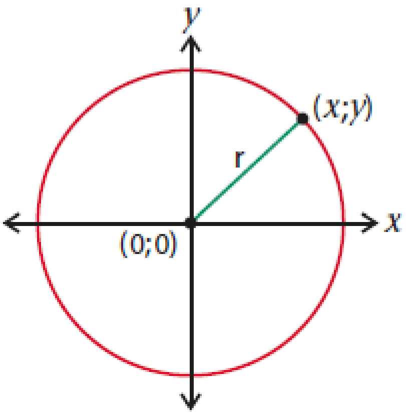
SESSION NO 2

EQUATION OF A CIRCLE

1. Circles that are centred at the origin

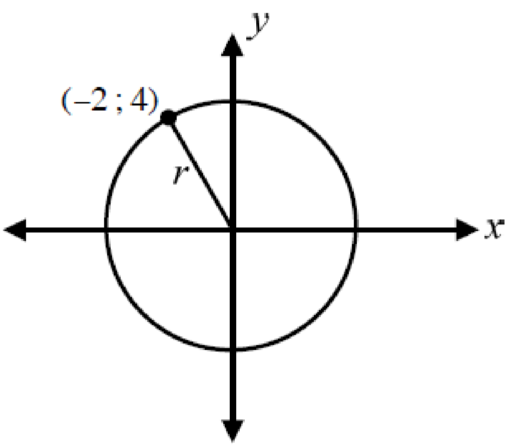
The equation of a circle with centre (0; 0) and radius *r* is:

$$x^2 + y^2 = r^2$$



WORKED EXAMPLE

Determine the equation of the circle with centre the origin passing through the point $(-2; 4)$.



Solution

$$x^2 + y^2 = r^2$$
$$\therefore (-2)^2 + (4)^2 = r^2$$

$$\therefore 4 + 16 = r^2$$
$$\therefore r^2 = 20$$

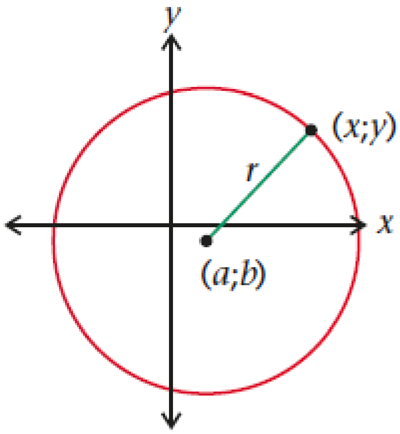
Therefore, the equation is;

$$x^2 + y^2 = 20$$

2. Circles that are NOT centred at the origin

The equation of the circle with centre (*a; b*) and radius *r* is:

$$(x - a)^2 + (y - b)^2 = r^2$$



WORKED EXAMPLE

Determine the equation of a circle with centre $(-2; 1)$ and radius 4 units.

Solution

Centre $(-2; 1)$

Radius $r = 4\ units$

$(x - a)^2 + (y - b)^2 = r^2$

$(x - -2)^2 + (y - 1)^2 = (4)^2$

$\therefore (x + 2)^2 + (y - 1)^2 = 16$

NOTE:

$(x - a)^2 + (y - b)^2 = r^2$ can be expressed in the form $\boldsymbol{x^2 + dx + y^2 + ey + f = 0}$.

Using $(x + 2)^2 + (y - 1)^2 = 16$ as an example, we expand the brackets and simplify the equation.

$(x + 2)^2 + (y - 1)^2 = 16$

$x^2 + 4x + 4 + y^2 - 2y + 1 = 16$

$\therefore x^2 + 4x + y^2 - 2y - 11 = 0$

It's so important that you know how to express $\boldsymbol{x^2 + dx + y^2 + ey + f = 0}$ in the form $(x-a)^2 + (y-b)^2 = r^2$.

This can be done by **completing squares** on the terms in x and y.

WORKED EXAMPLE

Given the circle with equation $2x^2 - 4x + 2y^2 + 32y + 78 = 0$.

a) Express the equation in the form $(x-a)^2 + (y-b)^2 = r^2$.
b) Hence, write down the coordinates of the centre of this circle and the radius.

Solution

a) $2x^2 - 4x + 2y^2 + 32y + 78 = 0$

$x^2 - 2x + y^2 + 16y + 39 = 0 \qquad (\boldsymbol{divide\ by\ 2})$

$(x-1)^2 - 1^2 + (y+8)^2 - 8^2 + 39 = 0$

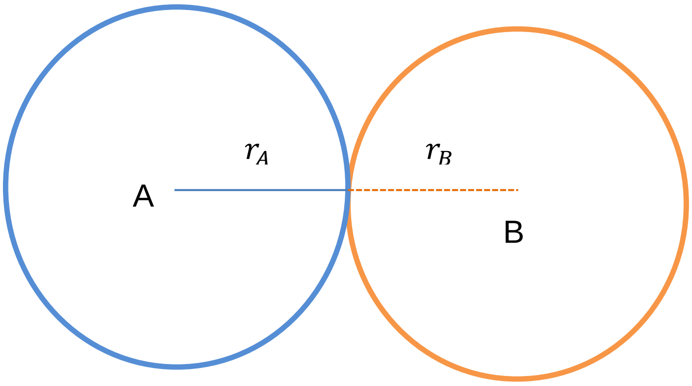
$\therefore (x-1)^2 + (y+8)^2 = 26$

b) Centre $(1; -8)$

Radius $= \sqrt{26}$ units

How to determine whether Two circles Touch, Intersect or Not

i. **Two circles touching:**

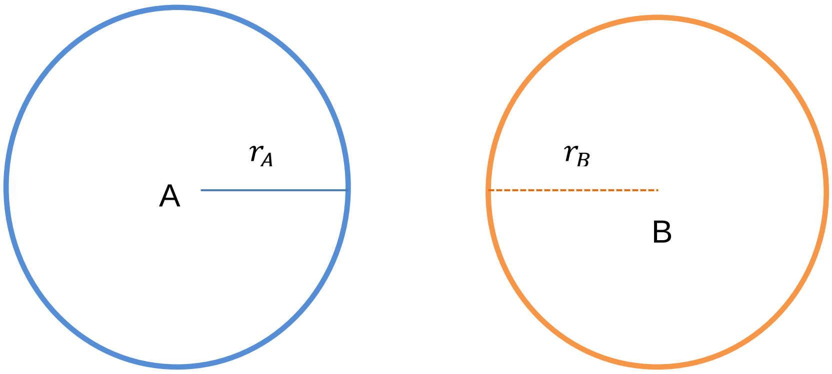


Circle centre A Circle centre B

The distance AB between the centres of the two circles equals the sum of the radii.

$\therefore AB = r_A + r_B$

ii. Two circles not touching or intersecting:

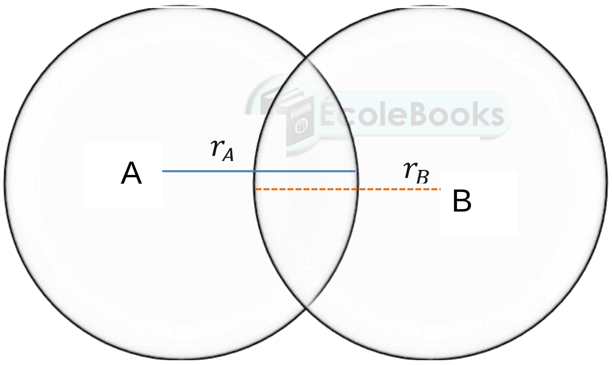


Circle centre A Circle centre B

The distance AB between the centres of the two circles is greater than the sum of the radii.

$\therefore AB > r_A + r_B$

iii. Two circles intersecting:



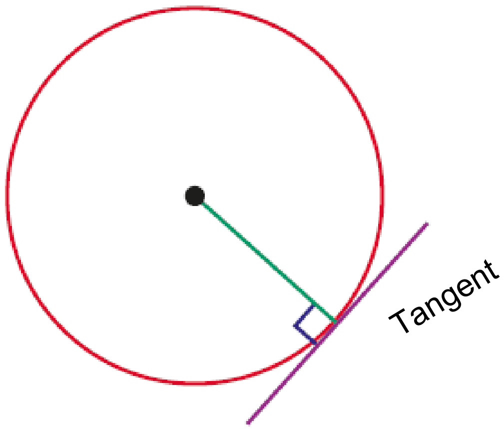
Circle centre A Circle centre B

The distance AB between the centres of the two circles is less than the sum of the radii.

$\therefore AB < r_A + r_B$

Equation of a tangent to the circle

The tangent to the circle is always perpendicular to the radius.

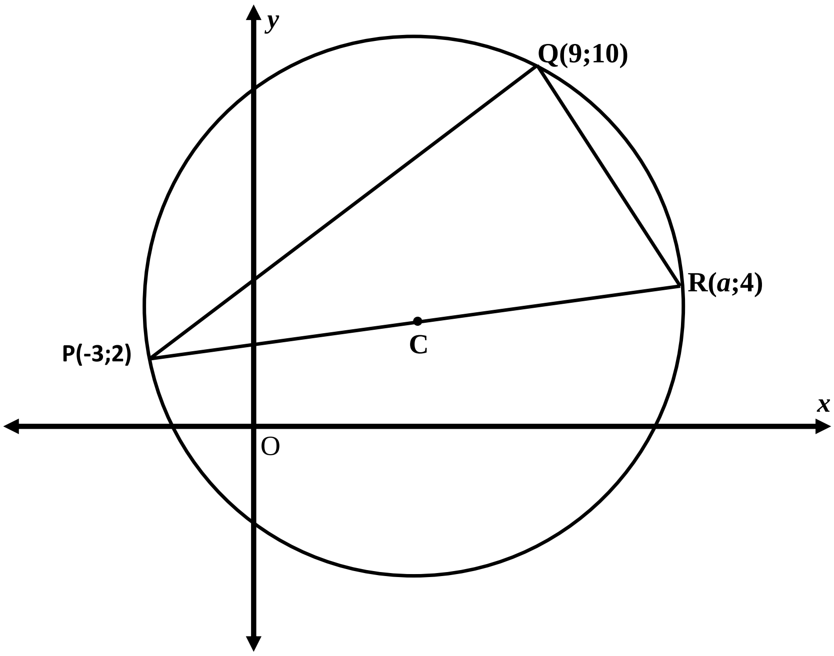


Remember the condition of perpendicular lines $m_1 \times m_2 = -1$.

Therefore $m_{tangent} \times m_{radius} = -1$

WORKED EXAMPLES

1. The points P(−3 ; 2), Q(9 ; 10) and R(*a* ; 4) lie on the circumference of the circle, as shown in the figure below. PR is a diameter of the circle with centre C.



1.1 Show that *a* = 13.

1.2 Determine the equation of the circle in the form:

$$(x-a)^2+(y-b)^2=r^2$$

Solutions

1.1 PQ ⊥ QR ∠ by diameter PR

$$m_{PQ} \times m_{QR} = -1$$
$$\frac{10-2}{9--3} \times \frac{4-10}{a-9} = -1$$
$$\frac{2}{3} \times \frac{-6}{a-9} = -1$$
$$\frac{-4}{a-9} = -1$$
$$-4 = -a + 9$$
$$\therefore a = 13$$

1.2

$$C\left(\frac{-3+13}{2}; \frac{2+4}{2}\right)$$
$$C(5;3)$$

Radius; using $P(-3;2)$ and the centre $C(5;3)$

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(-3-5)^2 + (2-3)^2 = r^2$$
$$65 = r^2$$
$$\therefore (x-5)^2 + (y-3)^2 = 65$$

2. The equations of two circles with centres A and B are given below:

Circle A: $(x-2)^2 + (y-3)^2 = 9$

Circle B: $(x-1)^2 + (y+1)^2 = 16$

Without solving for x and y, determine if the circles

- Intersect each other at 2 points.
- Touch each other.
- Do not intersect each other.

Show all your calculations.

Solution

Circle A: $(x-2)^2+(y-3)^2=9$

Circle B: $(x-1)^2+(y+1)^2=16$

Centres: A$(2;3)$ and B$(1;-1)$

Radii: $r_A = 3$ and $r_B = 4$

$r_A + r_B = 7$

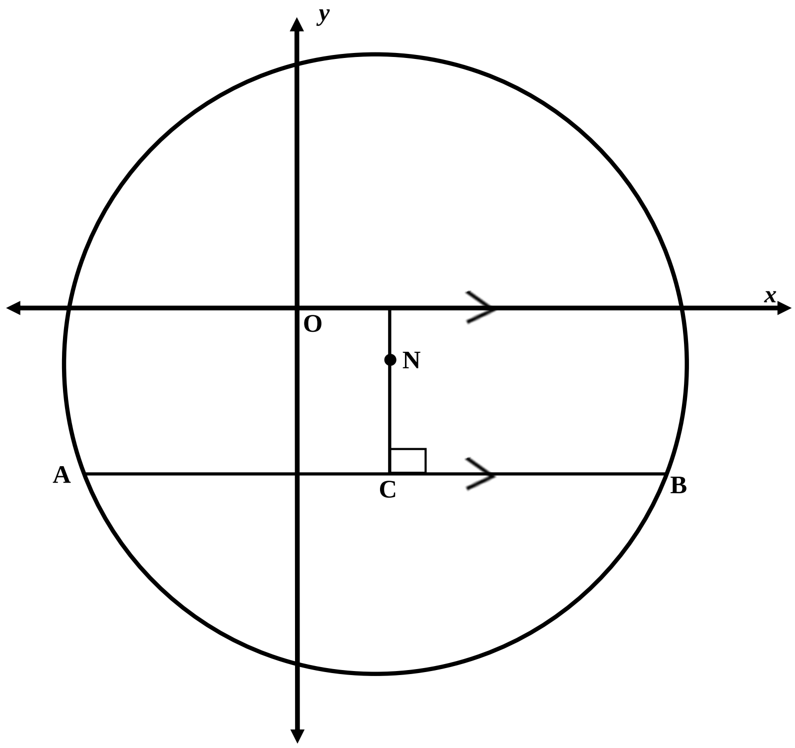
$AB = \sqrt{(2-1)^2 + (3--1)^2}$

$AB = \sqrt{17} = 4{,}12$

$AB < r_A + r_B$

∴The two circles intersect.

3. The figure below shows a sketch of the circle with centre N and equation $x^2 - 4x + y^2 + 2y = \frac{149}{4}$. Chord AB of circle N is parallel to the x-axis and lies below the x-axis. The length of AB is 12 units



3.1 Determine the coordinates of N.

3.2 Determine the radius of the circle.

3.3 Calculate the coordinates of A and B.

Solutions

3.1 $x^2 - 4x + y^2 + 2y = \frac{149}{4}$

$$(x-2)^2 - 2^2 + (y+1)^2 - 1^2 = \frac{149}{4}$$

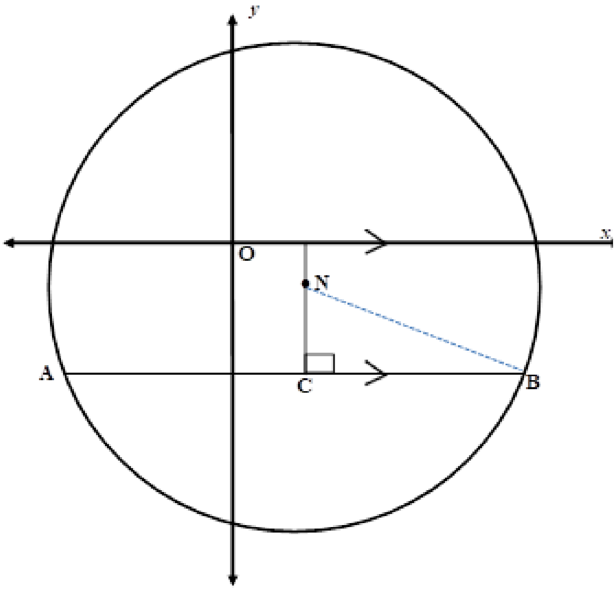
$$(x-2)^2 + (y+1)^2 = \frac{169}{4}$$

$N(2;-1)$

3.2 Radius:

$$r = \frac{13}{2}$$

3.3



$AC = CB$ line from centre of circle ⊥ chord AB

$CB = 6\,units$

$NB = \frac{13}{2}\ radius$

$N(2;-1)$ $C(2;y)$ $B(x;y)$ $A(-4;y)$

$NC = -1 - y$

$CB = x - 2$

$x - 2 = 6$

$x = 8$

Applying Pythagoras theorem on ΔNCB

$$(-1-y)^2 + 6^2 = \left(\frac{13}{2}\right)^2$$

$$(-1-y)^2 = \frac{25}{4}$$

$$-1 - y = \pm\frac{5}{2}$$

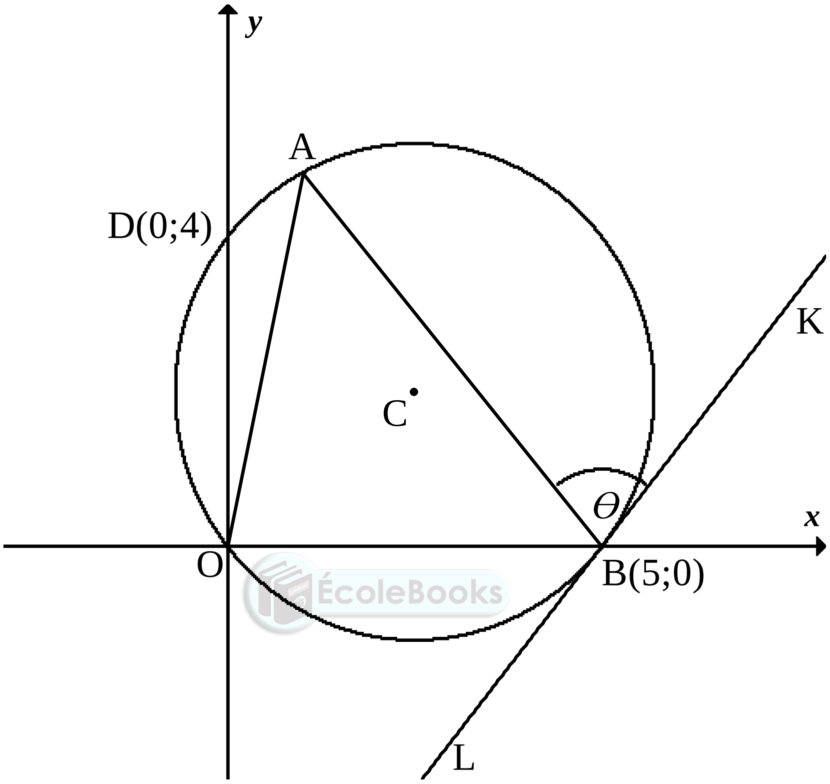
$y = -\frac{7}{2}$ OR $y = \frac{3}{2}$ $A\left(-4;-\frac{7}{2}\right)$ and $B\left(8;-\frac{7}{2}\right)$

ACTIVITY 2:

QUESTION 1

In the diagram below is the sketch of a circle with centre C$\left(\frac{5}{2};2\right)$.

The circle goes through the origin and cuts the *x*- and *y*-axis at B and D respectively. LBK is tangent to the circle at B. $A\hat{B}K = \theta = 78{,}69^0$.



1.1 Show by calculation that the equation of the circle is;

$$x^2 + y^2 - 5x - 4y = 0$$ (4)

1.2 Determine the equation of LBK in the form *y* = *mx* + *c*. (5)

1.3 Determine the gradient of OA correct to the nearest integer and hence the equation of OA. (3)

1.4 Calculate the coordinates of A. (5)

1.5 If there is another circle $\left(x + \frac{5}{2}\right)^2 + (y - 3)^2 = \frac{61}{4}$

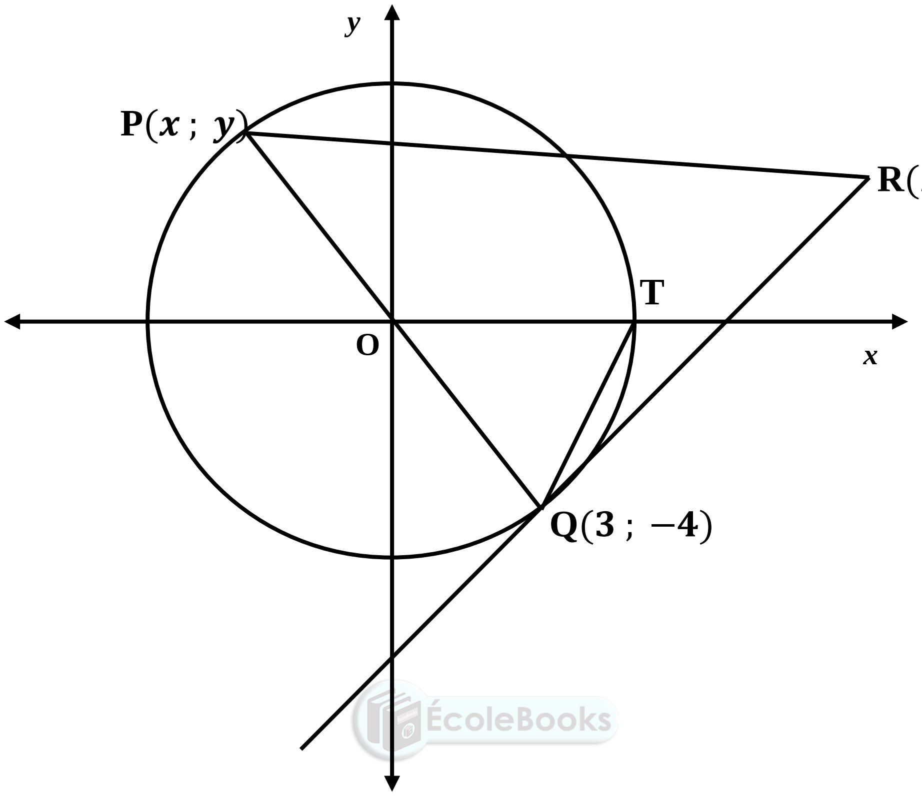
1.5.1 Show that the two circles intersect at two distinct points. (3)

1.5.2 Show by substitution that the two circles in fact intersect at O and A. (2)

[22]

QUESTION 2

In the figure below, the origin O is the centre of the circle. P$(x\ ;\ y)$ and Q$(3\ ;\ -4)$ are two points on the circle and POQ is a straight line. R is the point $(k\ ;\ 1)$ and RQ is a tangent to the circle. T is an x-intercept of the circle.



Determine:

2.1 the equation of the circle. (3)

2.2 the length of TQ. (Leave your answer in simplified surd form.) (3)

2.3 the equation of OQ. (3)

2.4 the coordinates of P. (2)

2.5 the equation of the circle with centre P, that passes through (0; 0) in the form $x^2 + y^2 = r^2$. (3)

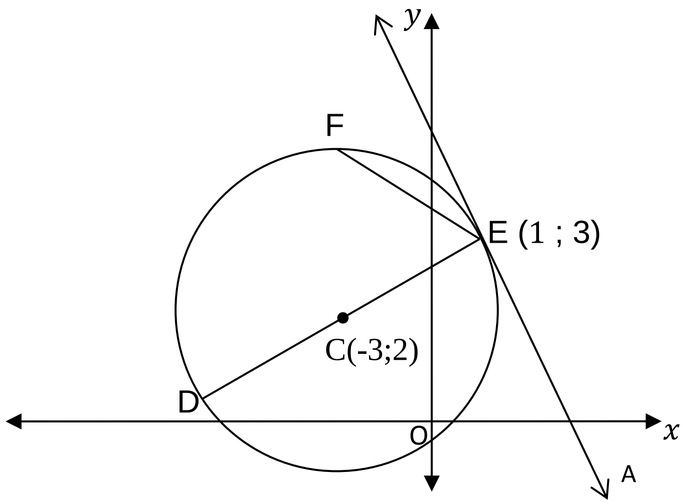
2.6 the equation of QR. (4)

2.7 the value of *k*. (3)

[21]

QUESTION 3

In the figure below, the centre C(−3 ; 2) of the circle lies on the diameter DE.
AE is a tangent to the circle at E (1; 3).



3.1 Calculate the length of the radius of the circle.
Leave the answer in surd form if necessary. (3)

3.2 Determine the equation of the tangent AE.
Give your answer in the form $y = mx + c$. (4)

3.3 Write down the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$ (2)

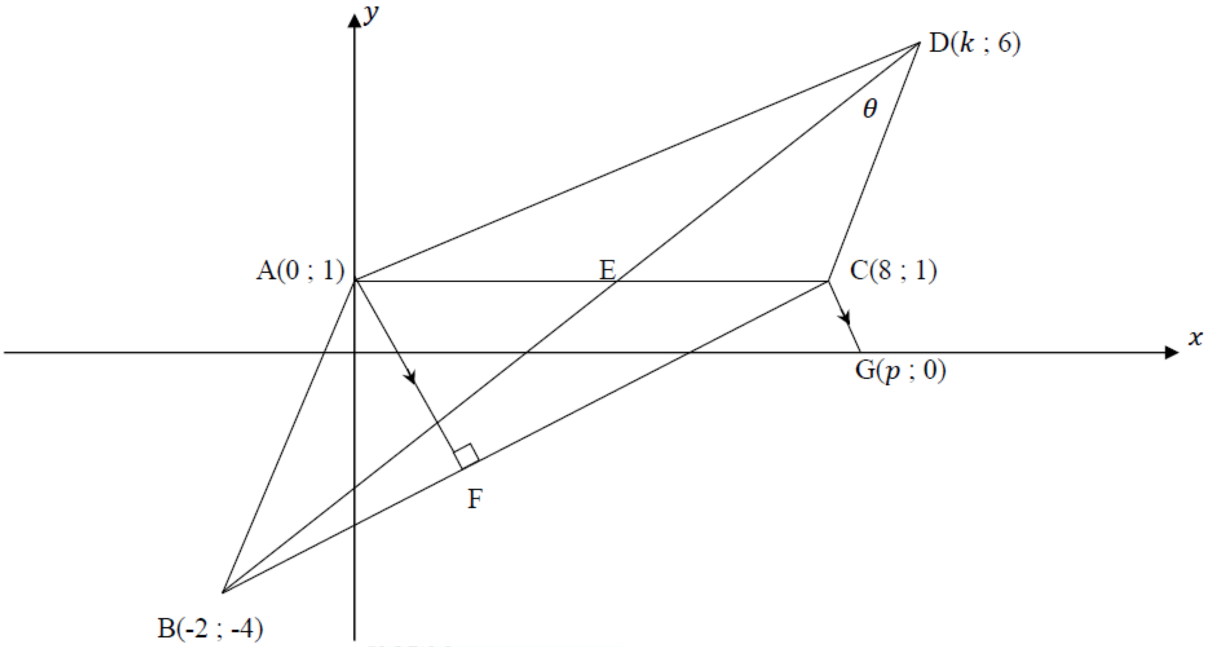
3.4. F is an endpoint of the chord EF. If the equation of chord EF is $y + x = 4$
determine the co-ordinates of F. (5)

[14]

HOME WORK

QUESTION 1

In the figure below, ABCD is a parallelogram with vertices A(0 ; 1), B(-2 ; -4), C(8 ; 1) and D(k ; 6). AF is perpendicular to BC and parallel to CG. E is a point of intersection of the diagonals of ABCD.



Determine:

1.1 the length of BC in simplest surd form. (2)

1.2 the gradient of BC. (1)

1.3 the equation of AF. (2)

1.4 the coordinates of E. (2)

3.5 the value of k. (2)

1.6 the value of p. (2)

1.7 the size of θ rounded off to one decimal place. (6)

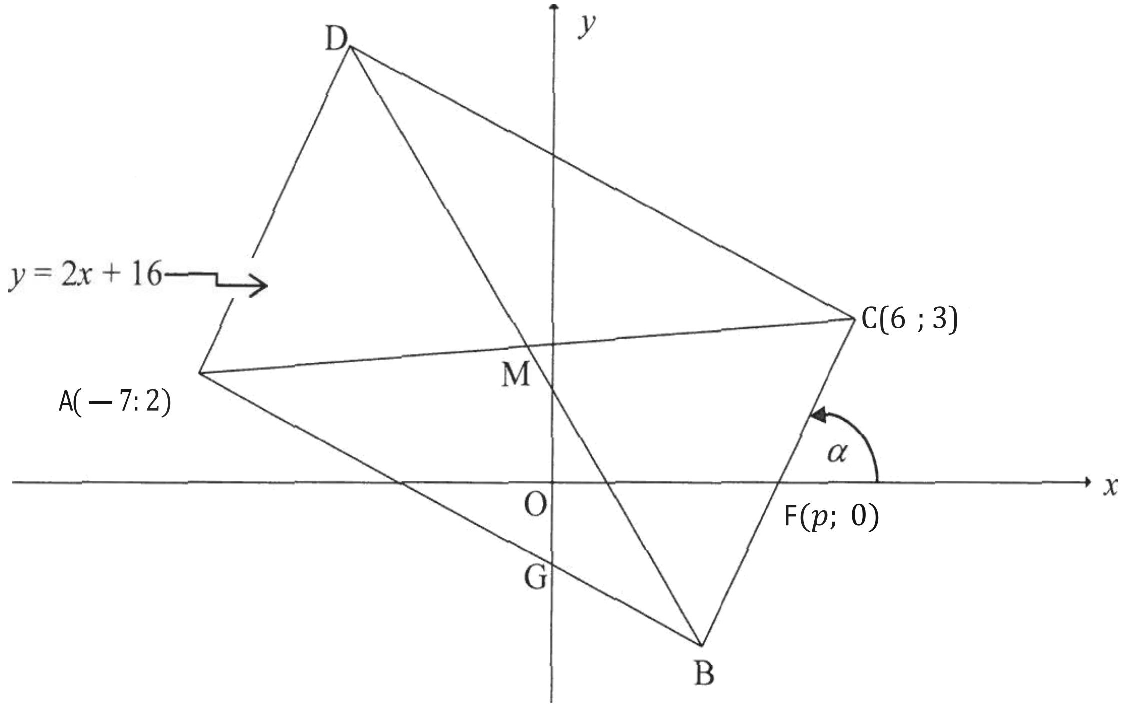
[17]

QUESTION 2

In the diagram, A(—7 ; 2), B, C(6 ; 3) and D are the vertices of rectangle ABCD.

The equation of AD is y = 2x + 16. Line AB cuts the y-axis at G. The x-intercept of line BC is P(p ; 0) and the angle of inclination of BC with the positive x-axis is α.

The diagonals of the rectangle intersect at M.



2.1 Calculate the coordinates of M. (2)

2.2 Write down the gradient of BC in terms of y. (1)

2.3 Hence, calculate the value of *p.* (3)

2.4 Calculate the length of DB. (3)

2.5 Calculate the size of a. (2)

2.6 Calculate the size of $O\hat{G}B$. (3)

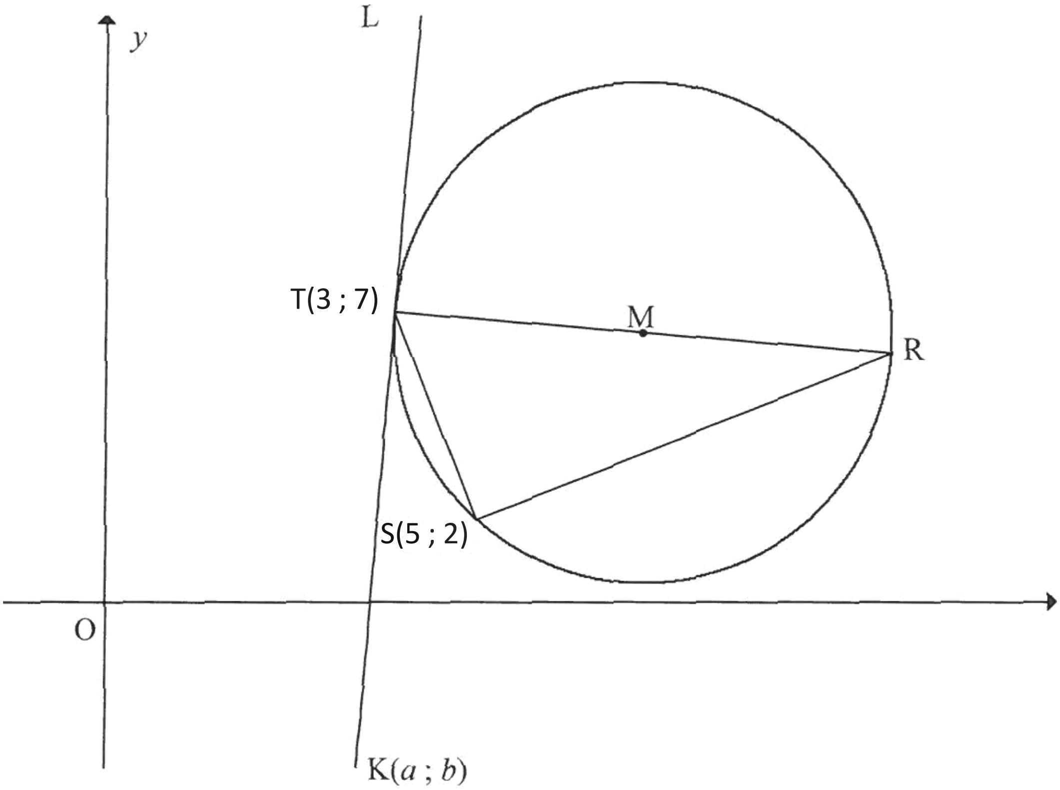
2.7 Determine the equation of the circle passing through points D, B and C in the form $(x-a)^2+(y-b)^2=r^2$. (3)

2.8 If AD is shifted so that ABCD becomes a square, will BC be a tangent to the circle passing through points A, M and B, where M is now the intersection of the diagonals of the square ABCD? Motivate your answer. (2)

[19]

QUESTION 3

In the diagram. M is the centre of the circle passing through T(3 ; 7), R and S(5 ; 2). RT isa diameter of the circle. K(u ; *b)* is a point in the 4" quadrant such that KTL is a tangent to the circle at T.



3.1 Give a reason why TSR — 90°. (1)

3.2 Calculate the gradient of TS. (2)

3.3 Determine the equation of the line SR in the form $y = mx + c$. (2)

3.4 The equation of the circle above is $(x-9)^2 + \left(y - 6\frac{1}{2}\right)^2 = 36\frac{1}{4}$

3.4.1 Calculate the length of TR in surd form. (2)

3.4.2 Calculate the coordinates or R. (3)

3.4.3 Calculate $\text{Sin}\,\hat{R}$. (3)

3.4.4 Show that b = 12a – 29. (3)

3.4.5 If TK = TR, calculate the coordinates of K. (6)

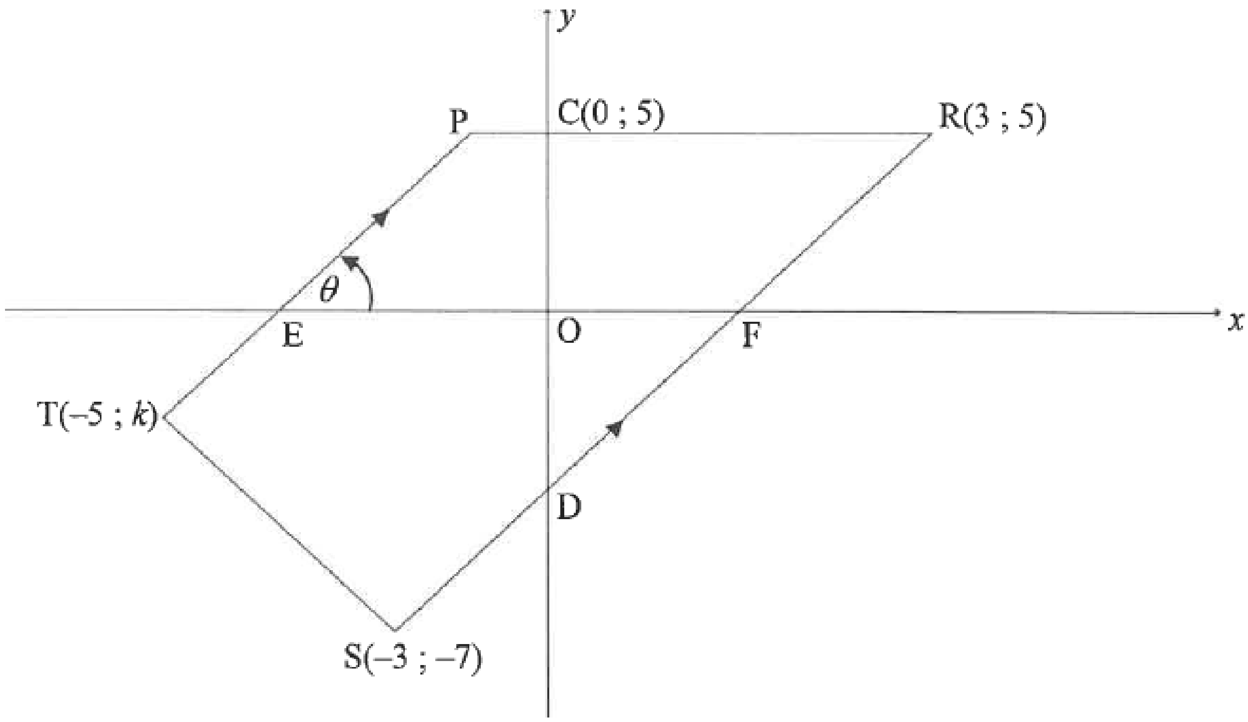
[23]

PAST EXAMINATION QUESTIONS

NOV 2019

QUESTION 1

In the diagram, P, R(3 ; 5), S(–3 ; –7) and T(–5 ; k) are vertices of trapezium PRST and PT || RS. RS and PR cut the y-axis at D and C(0 ; 5) respectively. PT and RS cut the x-axis at E and F respectively. $P\hat{E}F = \theta$.



1.1 Write down the equation of PR. (1)

1.2 Calculate the:

1.2.1 Gradient of RS. (2)

1.2.2 Size of θ. (3)

1.2.3 Coordinates of D. (3)

1.3 If it is given that $TS = 2\sqrt{5}$, calculate the value of k. (4)

1.4 Parallelogram TDNS, with N in the 4th quadrant, is drawn. Calculate the coordinates of N. (3)

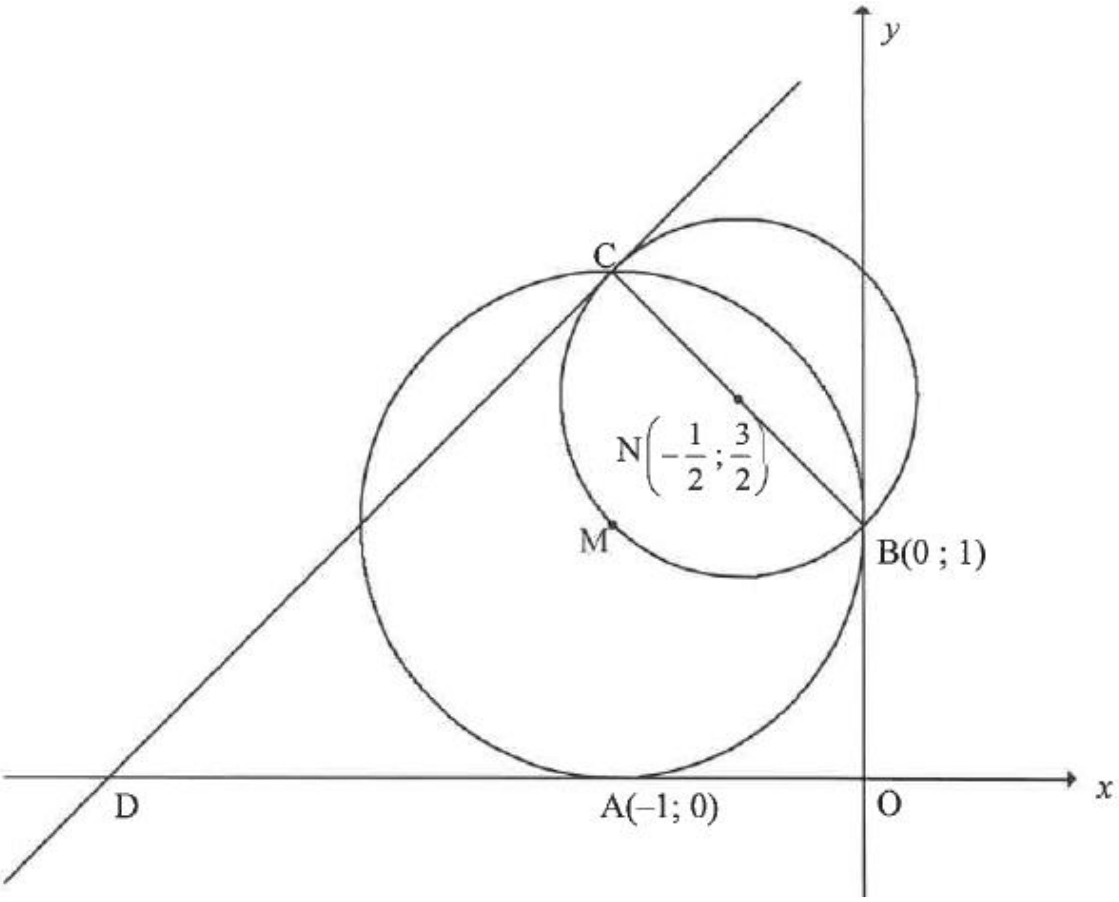
1.5 ΔPRD is reflected about the y –axis to form $\Delta P'R'D'$. Calculate the size of $R\widehat{D}R'$.

(3)

[19]

QUESTION 2

In the diagram, a circle having centre M touches the x–axis at A(–1 ; 0) and the y-axis a B(0 ; 1). A smaller circle, centred at $N\left(-\frac{1}{2}; \frac{3}{2}\right)$, passes through M and cuts the larger circl at B and C. BNC is a diameter of the smaller circle. A tangent drawn to the smaller circle a C, cuts the x-axis at D.



2.1 Determine the equation of the circle centred at M in the from $(x-a)^2+(y-b)^2=r^2$. (3)

2.2 Calculate the coordinates of C. (2)

2.3 Show that the equation of the tangent CD is $y-x=3$. (4)

2.4 Determine the values of t for which the line $y=x+t$ will NOT touch of cut the smaller circle. (3)

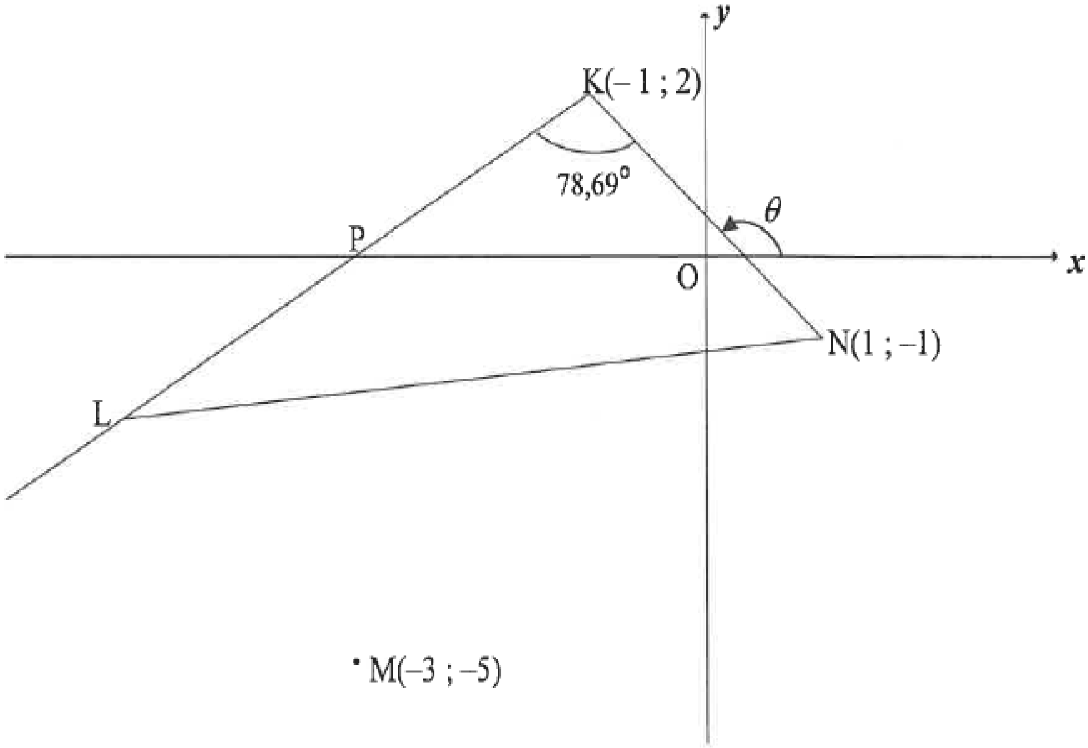
2.5 The smaller circle centred at N is transformed such that point C is translated along the tangent to D. calculate the coordinates of E, the new centre of the smaller circle. (3)

2.6 If it is given that the area of quadrilateral OBCD is $2a^2$ square units and $a>0$, show that $a=\frac{\sqrt{7}}{2}$ units. (5)

[20]

NOV 2018
QUESTION 3

In the diagram, K(– 1 ; 2), L and N(1 ; – 1) are vertices of ΔKLN such that $L\hat{K}N = 78,69°$. KL intersects the x-axis at P. KL is produced. The inclination of KN is θ. The coordinates of M are (–3 ; –5).



3.1 Calculate :

3.1.1 The gradient of KN. (2)

3.1.2 The size of θ, the inclination of KN. (2)

3.2 Show that the gradient of KL is 1. (2)

3.3 Determine the equation of the straight line Kl in the form $y = mx + c$. (2)

3.4 Calculate the length of KN. (2)

3.5 It is further given that KN = LM. (2)

3.5.1 Calculate the possible coordinates of L. (5)

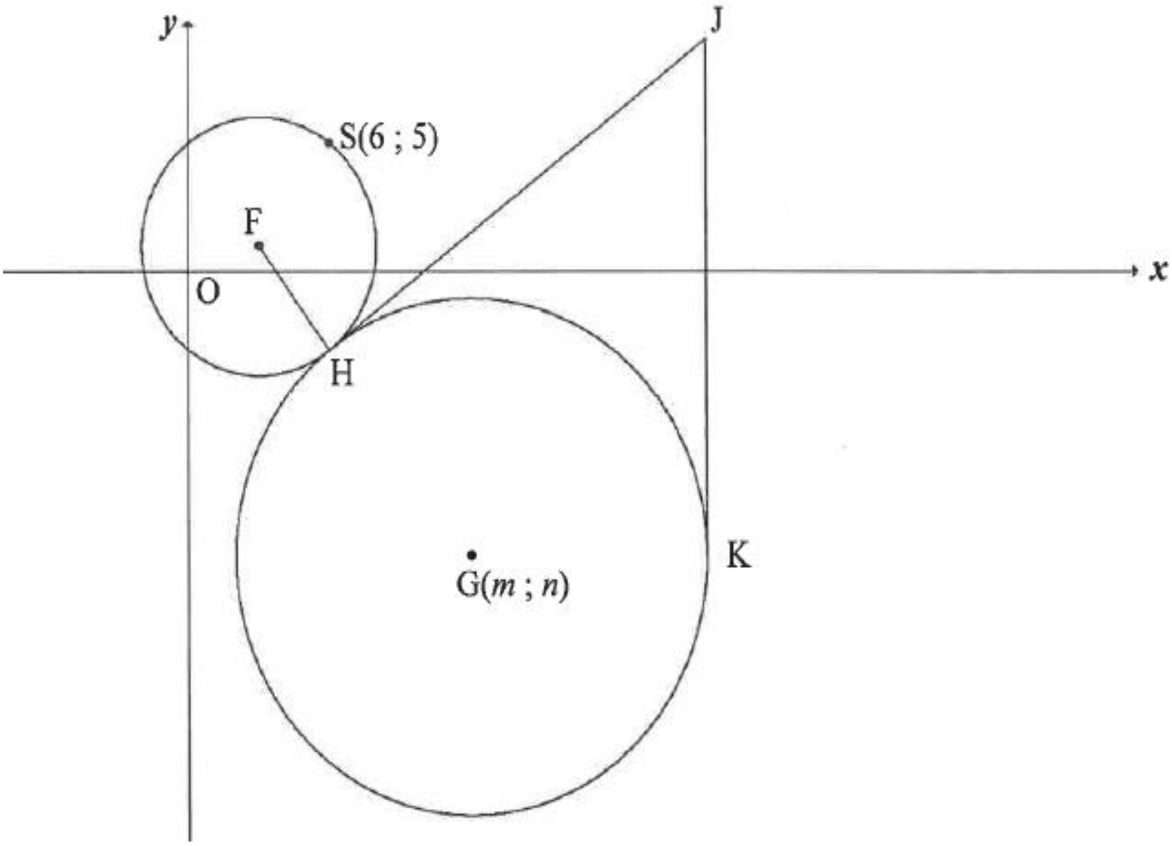
3.5.2 Determine the coordinates of L if it is given that KLMN is a parallelogram. (3)

3.6 T is a point on KL produced, TM is drawn such that TM = LM. Calculate the area of ΔKTN. (4)

[22]

QUESTION 4

In the diagram, the equation of the circle with centre F is $(x-3)^2+(y-1)^2=r^2$. S(6 ; 5) is a point on the circle with centre F. Another circle with centre G(m ; n) in the 4th quadrant touches the circle with centre F, at H such that FH : HG = 1 : 2. The point J lies in the first quadrant such that HJ is a common tangent to both these circles. JK is a tangent to the larger circle at K.



4.1	Write down the coordinates of F.	(2)
4.2	Calculate the length of FS.	(2)
4.3	Write down the length of HG.	(1)
4.4	Give a reason why JH = JK.	(1)
4.5	Determine:	

4.5.1	The distance FJ, with reasons, if it is given that JK = 20	(4)
4.5.2	The equation of the circle with centre G in terms of m and n in the form $(x-a)^2+(y-b)^2=r^2$	(1)
4.5.3	The coordinates of G, if it is further given that the equation of tangent JK is $x=22$	(7)

[18

<u>SESSION NO: 3</u>

<u>TOPIC: FUNCTIONS AND GRAPHS</u>

In this session we briefly revise the concept of a functions, which was studied in grade 10 and 11.

At the end of the lesson, a learner should be able to do the following:

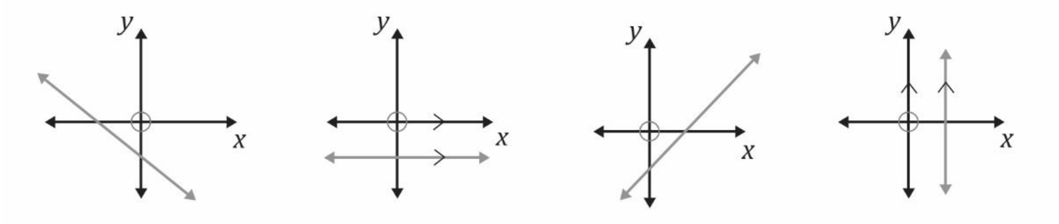
- Identify different types of graphs in terms of equation and/or sketch
- Draw the graph given the equation of the graph
- Determine the equation of the function or graph given the sketch and some points on the sketch
- Determine the co-ordinates of the turning point of the parabola
- Determine the equation of the asymptote of exponential function
- Determine the equation of the asymptotes and lines of symmetry of hyperbolic functions
- Determine the intercepts of each function with axes
- Determine the domain and range of all the functions
- Determine coordinates of points of intersections of different functions.

1.1 **Straight line**

General representation or equation

$y = ax + q$ or $y = mx + x$. a or m is the gradient and q or c is the $y-$intercept

Also note the shape of the following linear functions



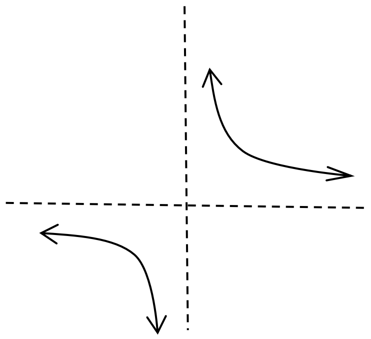
a < 0	*a* = 0	*a* > 0	*a* is undefined
q < 0	*y* = *q*	*q* < 0	there is no *q*-value

Domain and range is $x \in \Re$ and $y \in \Re$ respectively

1.2 Hyperbola

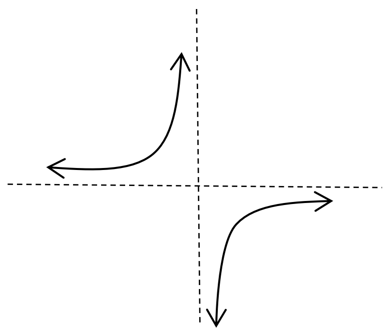
- General representation or equation
- $y = \frac{a}{x}$ or $xy = a$ $\quad y = \frac{a}{x} + q$ or
- $y = \frac{a}{x+p} + q$

- $a > 0$



Dotted lines are asymptotes

- $a < 0$



Dotted lines are asymptotes

- q is the vertical translation
- p is the horizontal translation
- For $y = \frac{a}{x}$, $p = 0$ and $q = 0$. The **vertical** asymptote is $x = 0$ and the **horizontal** asymptote is $y = 0$. The **axis of symmetry** are $y = x$ (Positive) and $y = -x$ (Negative)
- **Domain** is $x \neq 0, x \in \Re$ and **Range** is $y \neq 0,\ y \in \Re$
- For $y = \frac{a}{x} + q$, $p = 0$. The **vertical** asymptote is $x = 0$ and the **horizontal** asymptote is $y = q$. The **axis of symmetry** are $y = x + q$ (Positive) and $y = -x + q$ (Negative).
- **Domain** is $x \neq 0,\ x \in \Re$ and **Range**, $y \neq q,\ y \in \Re$

- For $y = \frac{a}{x-p} + q \Rightarrow (y-q)(x-p) = a$, the **vertical** asymptote is $x = p$ and the **horizontal** asymptote is $y = q$. The **axis of symmetry** is $y = \pm(x-p) + q$.
- **Domain** is $x \neq -p,\ x \in \Re$ and **Range**, $y \neq q,\ y \in \Re$

- For $y = \frac{a}{x+p} + q \ \Rightarrow (y-q)(x+p) = a$, the **vertical** asymptote is $x = -p$ and the **horizontal** asymptote is $y = q$. The **axis of symmetry** is $y = \pm(x+p) + q$.
- **Domain** is $x \neq -p,\ x \in \Re$ and range is $y \neq q,\ y \in \Re$

Worked example 1

Given : $f(x) = \frac{3}{x-2} + 1$

(a) Write down the equations of the asymptotes of f.

(b) Determine coordinates of B, the *x*-intercept of *f*.

(c) Determine the coordinates of D, the y-intercept of f

(d) Determine the domain and the range of f

(e) Determine the decreasing and increasing functions of the axes of symmetry of f

(f) Draw the sketch graph of f

Solution:

(a) Vertical asymptote is

$x-2=0$

$x=2$

Horizontal asymptote is $y=1$

(b) $x-$intercept $\Rightarrow y=0$

$$0=\frac{3}{x-2}+1$$

$-1(x-2)=3$

$-x+2=3 \Rightarrow -x=1$

$\therefore x=-1$

(c) $y-$intercept $\Rightarrow x=0$

$$y=\frac{3}{-2}+1=\frac{3-2}{-2}=-\frac{1}{2}$$

(d) Domain is $x\neq 2; x\in\Re$

Range is $y\neq 1; y\in\Re$

(e) Axes of symmetry are:

$$y=\pm(x-2)+1$$

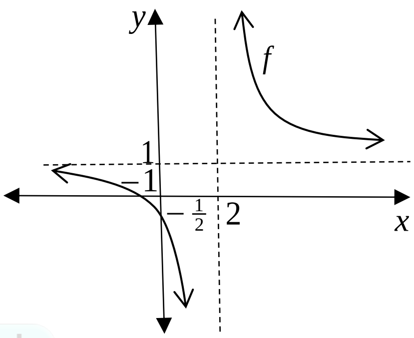
$=x-2+1\ or\ y=-x+2+1$

$y=x-1$

Increasing (positive gradient) or

$y=-x+3$ Decreasing (Negative gradient)

(f)



1.3 Parabola

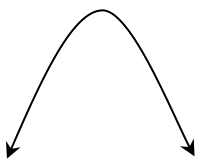
General representation or Equation

$y = ax^2$ or $y = ax^2 + q$ or $y = a(x+p)^2 + q$ or

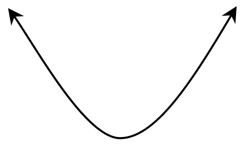
$y = ax^2 + bx + c$

Important Deductions

for $a < 0$



for $a > 0$



- For $y = ax^2$, $p = 0$ and $q = 0$, the **turning point** is $(0;0)$ and **y-intercept** is y = 0

  The **domain** is $x \in \mathbb{R}$ and the **range** is $y \geq 0; y \in \mathbb{R}$ if $a > 0$ or $y \leq 0; y \in \mathbb{R}$ if $a < 0$

- For $y = ax^2 + q$, $p = 0$, the **turning point** is $(0;q)$ and **y-intercept** is y = q

  The **domain** is $x \in \mathbb{R}$ and the **range** is $y \geq q; y \in \mathbb{R}$ if $a > 0$ or $y \leq 0; y \in \mathbb{R}$ if $a < 0$

- For $y = a(x+p)^2 + q$, the **turning point** is $(-p;q)$ and **y-intercept** is $y = a(p)^2 + q$

  The **domain** is $x \in \mathbb{R}$ and the **range** is $y \geq q; y \in \mathbb{R}$ if $a > 0$ or $y \leq 0; y \in \mathbb{R}$ if $a < 0$

- For $y = ax^2 + bx + c$, the **turning point** is $\left(\frac{-b}{2a}; \frac{4ac - b^2}{4a}\right)$ and **y-intercept** is y = c

  The **domain** is $x \in \mathbb{R}$ and the **range** is $y \geq \frac{4ac - b^2}{4a}; y \in \mathbb{R}$ if $a > 0$ or $y \leq \frac{4ac - b^2}{4a}; y \in \mathbb{R}$ if $a < 0$

- The **roots or *x*-intercepts** are determined by equating y to zero and solve for x.

FINDING THE EQUATION OF PARABOLA

A. Steps to find the equation of a parabola when given two $x-$ intercepts and any point$(x;y)$ on the graph.

Step:1.

- Use the formula $y = a(x - x_1)(x - x_2)$
- Substitute the 2 $x - intercepts$ into x_1 and x_2 then substitute the point $(x;y)$ into $y = a(x - x_1)(x - x_2)$ where there is $(x;y)$.

Step:2.

- Solve for a.

Step:3.

- Substitute a, x_1 and x_2 back into the original equation $y = a(x - x_1)(x - x_2)$ then simplify and leave it in standard form $y = ax^2 + bx + c$

B. **Steps to finding the equation of a parabola when given the Turning Point$(p;q)$ and any point on the graph** $(x;y)$.

Step:1.

- Substitute the **Turning Point**$(p;q)$ into the equation: $y = a(x - p)^2 + q$

Step:2.

- Substitute the point $(x;y)$ into the equation: $y = a(x - p)^2 + q$,where $there\ is\ x\ and\ y$.

Step:3.

- Solve for a.

Step:4.

- Substitute all the values: a,p ,and q.

Step:5.

- Write the equation in the format the question requires it in.
  Either: $y = ax^2 + bx + c$ or $y = a(x - p)^2 + q$

C. **Steps to finding the equation of a parabola when given the $y-$ intercept and any two points on the graph.**

Step:1.

- Find the y-intercept from the given information, the c in $y = ax^2 + bx + c$.

Step:2.

- Substitute the two coordinates into $y = ax^2 + bx + c$.
- Derive two equations.
- Solve the equations simultaneously for a and b.

Finding the equation of the hyperbola

Step:1.

- Identify the asymptotes (the vertical and horizontal asymptotes), p and q.

Step:2.

- Use a point or coordinate on the graph to determine the value of a.

AXIS OF SYMMETRY

- The axis of symmetry is the line that divides a graph into two equal halves.
- The hyperbola has two axis of symmetry lines.

NB:

The axis of symmetry lines pass the point of intersection of the asymptotes(p;q).

Finding the equation of an exponential graph $y = a.b^{x-p} + q$

Step:1.

- When we have two points on the graph, we substitute the point on the y-axis (where x=0) to calculate the value of a.

Step:2.

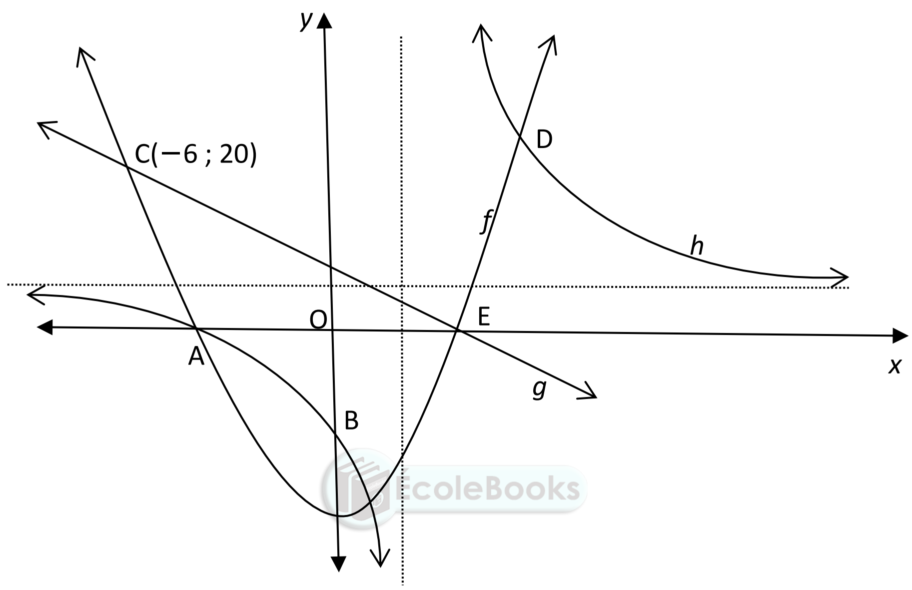
- After finding a we substitute a back into the original equation then substitute with it the second point to solve for b.

Worked example 2

Sketched below are the graphs of: $g(x) = -2x + 8$; $f(x) = x^2 + k$ and

$h(x) = \dfrac{6}{x-2} + 1$

A and B are the *x*- and *y* - intercepts of *h* respectively, C (−6; 20) and E are the points of intersection of *f* and *g* .



(a) Calculate the coordinates of A, B and E.

(b) Show that the value of $k = -16$

(c) Determine the domain and the range of *f*

(d) Write down the values of *x* for which $g(x) - f(x) \geq 0$

(e) Determine the equation of the symmetry axis of *h* if the gradient is negative.

(f) Write down the range of *s*, if *s*(*x*) = *f*(*x*) + 2.

(g) Write down the range of *t*, if *t(x)* = *h(x)* + 2

Solutions

NB: To answer the above questions you need to identify all the functions in order to apply the deductions indicated above.

(a) A and B are *x* and *y* intercepts of *g* respectively.

at A, $y = 0 \therefore \frac{6}{x-2} + 1 = 0$

$6 = -x + 2$

$4 = -x$

$\therefore x = -4$

Thus A$(-4;0)$

at B, $x = 0 \therefore y = \frac{6}{-2} + 1$

$y = -3 + 1$

$\therefore y = -2$

Thus B$(0;-2)$

E is the x- intercept of the straight line and the parabola. It is easy and straight forward to use the
equation of the straight line to get the coordinates of E.

At E, $y = 0; \ \therefore 0 = -2x + 8$

$2x = 8$

$x = 4$

Thus E$(4;0)$

b) C(-6; 20) is on *f* and *g* , substituting the
into

$y = x^2 + k \Rightarrow 20 = (-6)^2 + k$

$\therefore 20 - 36 = k$

$k = -16$

c) Domain is $x \in \mathbb{R}$

Range is $y \geq -16; \ y \in \mathbb{R}$

d) These are values of *x* for which the graph of *g* and *f* intersect or *f* is below *g* .

It is from C(-6 ; 20) and E(4 ; 0)

That is $-6 \leq x \leq 4$

e) For negative gradient, $y = -(x-2) + 1$

$y = -x + 2 + 1$

$y = -x + 3$

f) + 2 implies the value of *p* is increased by 2

The range of *s* is $y \geq -16 + 2$

$y \geq 14$

g) + 2 implies the value of *p* is increased by 2

The range of t is $y \neq 1 + 2; \ y \in \mathbb{R}$

$y \neq 3; \ y \in \mathbb{R}$

1.4 Exponential

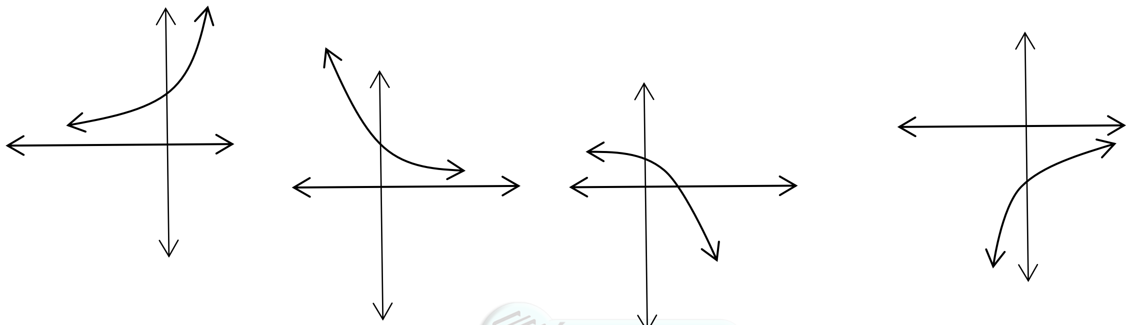
General representation or Equation:

$y = ab^x$ or $y = ab^x + q$ or $y = ab^{x+p} + q$

The restriction is $b > 0;\ b \neq 1$

Important Deductions

$a > 0 \text{ and } b > 1$ for $a > 0 \text{ and } 0 < b < 1$ for $a < 0 \text{ and } b > 1$ for $a < 0 \text{ and } 0 < b < 1$



- or $y = ab^x$, the **asymptote** is y = 0 and the **y-intercept** is $y = a$
- For $y = ab^x + q$, the **asymptote** is y = *q* and **y-intercept** is y = *a* + *q*
- For $y = ab^{x+p} + q$, the **asymptote** is y = *q* and **y-intercept** is $y = ab^p + q$

Worked example 3

Given: $f(x) = 3^{-x+1} - 3$

(a) Write $f(x)$ in the form $y = ab^x + q$

(b) Draw the graph of f showing all the intercepts with the axes and the asymptote.

(c) What is the domain and the range of f ?

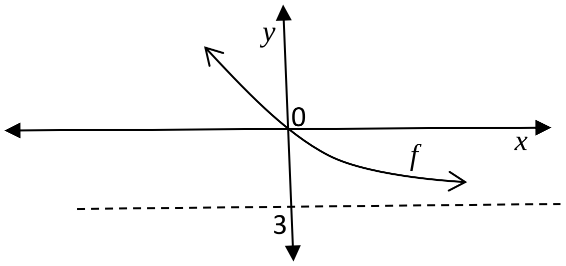
Solutions:

(a) $y = 3^{-x+1} - 3 = 3^{-x}.3 - 3 = 3.3^{x} - 3 = 3\left(\frac{1}{3}\right)^{x} - 3$

(b) The asymptote is $y = -3$, x-intercept, y = 0, i.e. $3\left(\frac{1}{3}\right)^{x} - 3 = 0$

$3\left(\frac{1}{3}\right)^{x} = 3 \Rightarrow \left(\frac{1}{3}\right)^{x} = 1 = \left(\frac{1}{3}\right)^{0}$

$\therefore x = 0$

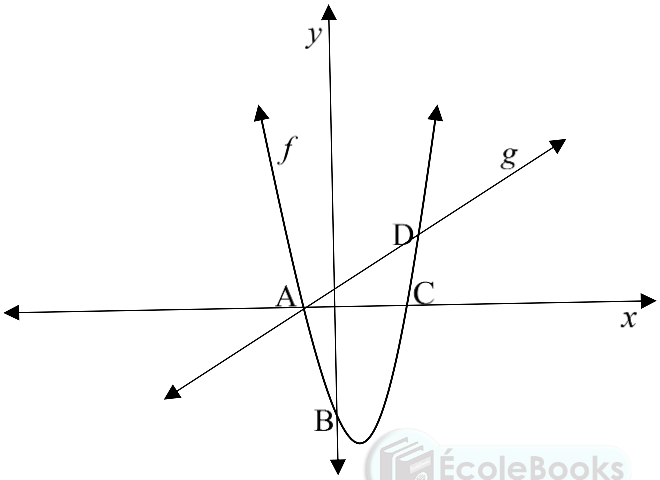


(c) Domain is $x \in \mathbb{R}$ and range is $y > -3$; $y \in \mathbb{R}$.

ACTIVITY 1: FUNCTIONS AND GRAPHS

Question 1

The graphs of $f(x) = 2x^2 - x - 3$ and $g(x) = x + 1$ are sketched below.



1.1 What is the gradient of *g*

1.2 Determine the coordinates of A and C.

1.3 Determine the coordinates of D

1.4 What is the y-intercept of *f* ?

1.5 Determine the axis of symmetry of *f.*

1.6 Determine the coordinates of the turning point of *f*.

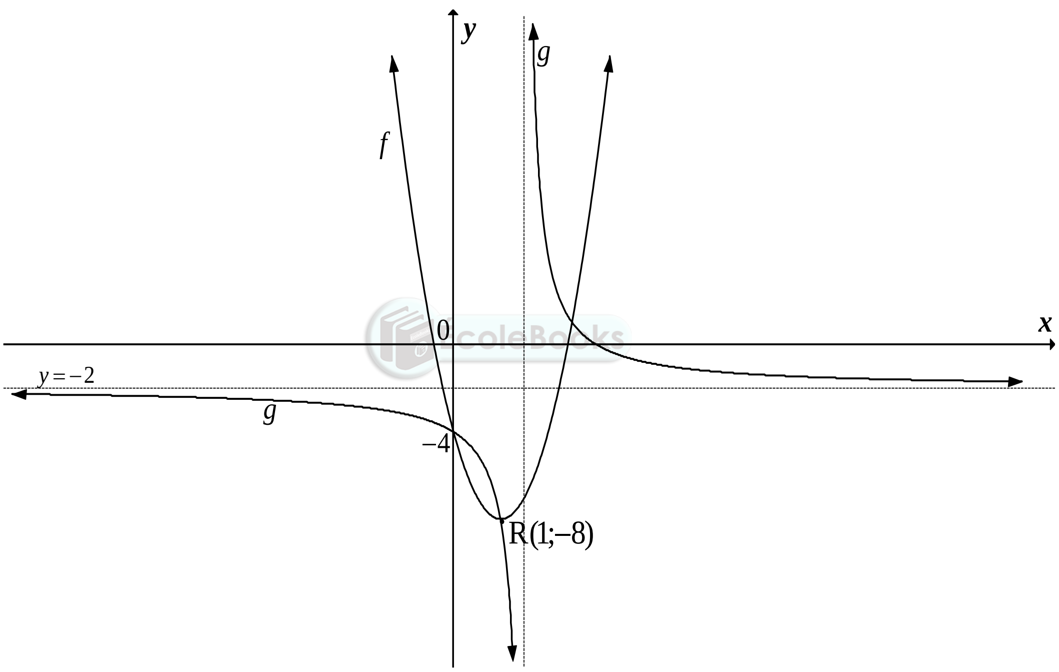
1.7 What is the domain and the range of *f* ?

Question 2

The graphs of the functions $f(x) = a(x+p)^2 + q$ and $g(x) = \dfrac{k}{x+t} + d$ are sketched below.

Both graphs cut the *y*-axis at -4. One of the points of intersection of the graphs is

R(1 ; –8), which is also the turning point of *f*. The horizontal asymptote of *g* is $y = -2$.



2.1 Calculate the values of *a*, *p* and *q.*

2.2 Calculate the values of *k*, *t* and *d*.

2.3 Determine the value(s) of *x* in the interval $x \leq 1$ for which $g(x) \geq f(x)$.

2.4 Determine the domain and the range of *f*.

2.5 Write down equations for the axes of symmetry of *g*, both with negative and positive gradient.

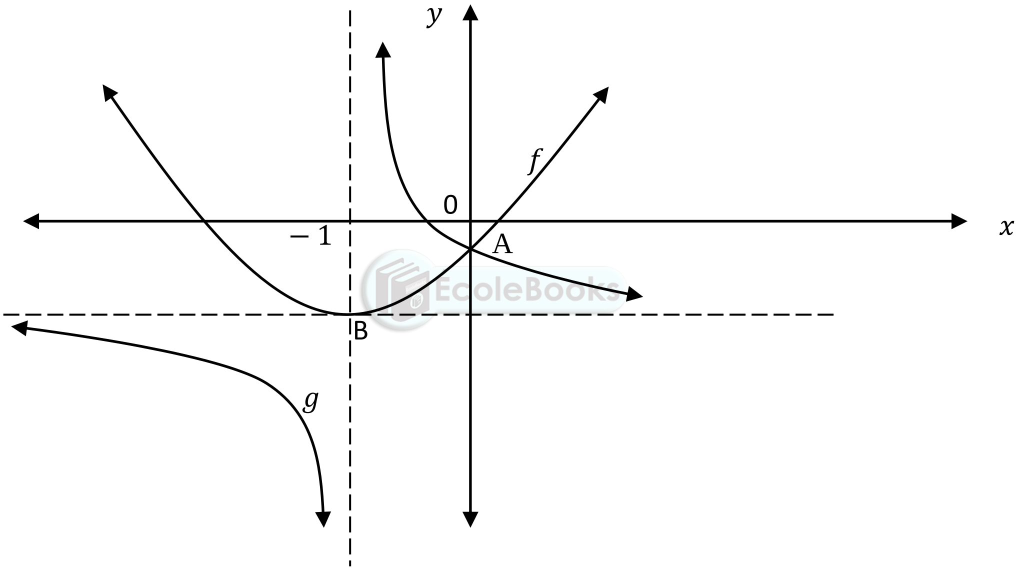
Question 3

3.1 Draw sketch graphs of $g(x) = 2^{-x} - 2$ and $h(x) = -\frac{3}{x+1} - 2$ on the same set of axes.

3.2 Show all the intercepts with the axes and the and asymptotes.

Question 4

The graphs of $f(x) = x^2 + 2x - 3$ and $g(x) = \frac{a}{x+p} + q$ are drawn below. A is the $y-$ intercept of both f and g. The horizontal asymptote of g is also a tangent to f at B, the turning point of f. The equation of the vertical asymptote of g is $x = -1$.



4.1 Write down the coordinates of B

4.2 Determine the equations of the asymptotes of g.

4.3 Write down the coordinates of A.

4.4 Determine the equation of g.

4.5 Determine the equations of axes of symmetry of .

4.6 Write down the range of $f(x)$ and that of $-f(x)$.

Question 5

Given: $f(x) = 2^{x+1} - 8$

5.1 Write down the equation of the asymptote of *f*.

5.2 Sketch the graph of *f*. Clearly indicate ALL intercepts with the axes as well as the asymptote.

SESSION NO: 4

TOPIC: FUNCTIONS AND INVERSE FUNCTIONS

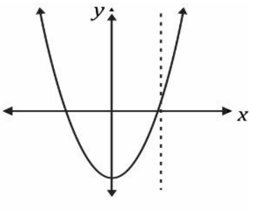
1.1 Functions

- **A function** is a relationship between x and y , where for every x -value there is only y-value.
- One way to decide whether or not a graph represent a function is to use the **vertical line test**.
- If any line drawn parallel to the y-axis cuts the graph only once, then the graph represents a function.

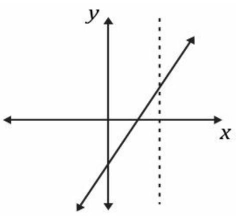
Worked example 1

Are the following graphs representing a function or not?

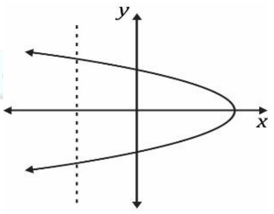
Graph 1



Graph 2



Graph 3



Solutions:

Graph 1 and Graph 2 are functions.

Graph 3 is not a function because the vertical line cuts the graph twice. i.e. for some *x*-value on the graph, there are two *y*-values.

1.2 Inverse Functions

- **Inverse of a function** is obtained by interchanging x and y of the original function. i.e. If $y = ax + b$ then its inverse is $x = ay + b$. Making y the subject of the formula, $y = \frac{1}{a}(x-b)$ or $y = \frac{x}{a} - \frac{b}{a}$
- The inverse of a function is the mirror image(Reflection) of the function along the line $y = x$.

- Notation of the inverse is f^{-1}.

Worked example 2

1. Determine the inverse of the following functions in the form $y =$........
   (a) $f(x) = 2x + 3$
   (b) $g(x) = 3x^2$
2. Restrict the domain of $g(x) = 3x^2$ such that its inverse will also be a function

Solutions:

(a) step 1: Interchange x and y i.e. $x = 2y + 3$

step 2: Make y the subject of the formula:

$$2y + 3 = x$$

$$2y = x - 3$$

$$y = \frac{x-3}{2} \quad or \quad y = \frac{1}{2}x - \frac{3}{2}$$

(b) step 1: interchange *x* and *y*, i.e. $x = 3y^2$

step 2: Make *y* the subject of the formula:

$$3y^2 = x$$

$$y^2 = \frac{x}{3}$$

$$\therefore y = \pm\sqrt{\frac{x}{3}}$$

(2) $y = ax^2$, The domain can be restricted to either $x \leq 0; x \in \mathbb{R}$ or $x \geq 0;\ x \in \mathbb{R}$

1.3 Logarithmic Function

- $y = \log_b x$ is a logarithmic function.
- $y = \log_b x$ Reads "*y* is equal to log x base *b*"
- The logarithmic function is only defined if $b > 0$, $b \neq 1$ and $x > 0$
- An exponential equation can be written as a logarithmic equation and vice versa. The base of the exponential equation becomes the base of the logarithmic equation.

Worked example 3

Write each of the following exponential equations as logarithmic equations

(a) $2^6 = 64$

(b) $5^3 = 125$

Solutions

(a) $2^6 = 64$
$6 = \log_2 64$

(b) $5^3 = 125$
$3 = \log_5 125$

NB: The inverse of the exponential function $y = a^x$ is $x = a^y$.

Making *y* the subject of the formula, $a^y = x$

$$\log a^y = \log x$$
$$y \log a = \log x$$
$$y = \frac{\log x}{\log a} = \log_a x$$

Thus logarithmic function is the inverse of exponential function. i.e. If $f(x)=a^x$ then $f^{-1}(x)=\log_a x$

Worked example 4

Given: *f(x)* = 3^x

(a) Determine f^{-1} in the form y =……

(b) Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y=x$ on the same set of axes.

(c) Write the domain and range of $f(x)$ and $f^{-1}(x)$

Solutions:

(a) The inverse of $y=3^x$ is $x=3^y$. Changing $x=3^y$ to logarithmic form, it becomes $y=\log_3 x = f^{-1}$

Using table method

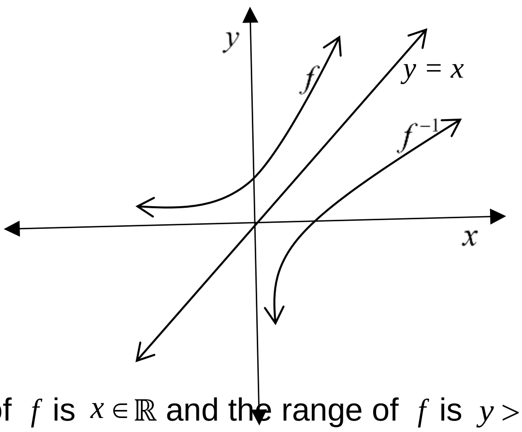
$f(x)$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

$f^{-1}(x)$

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$f^{-1}(x)$	-2	-1	0	1	2

(b)



(c) The domain of f is $x \in \mathbb{R}$ and the range of f is $y>0, y \in \mathbb{R}$

The domain of f^{-1} is $x>0, x \in \mathbb{R}$ and the range of f^{-1} is $y \in \mathbb{R}$

ACTIVITY 1: FUNCTIONS AND INVERSE FUNCTIONS

Question 1

Consider the functions: $f(x)=2x^2$ and $g(x)=\left(\frac{1}{2}\right)^x$

1.1 Restrict the domain of f in one specific way so that the inverse of f will also be a function.

1.2 Hence draw the graph of your new function *f* and its inverse function f^{-1} on the same set of axes.

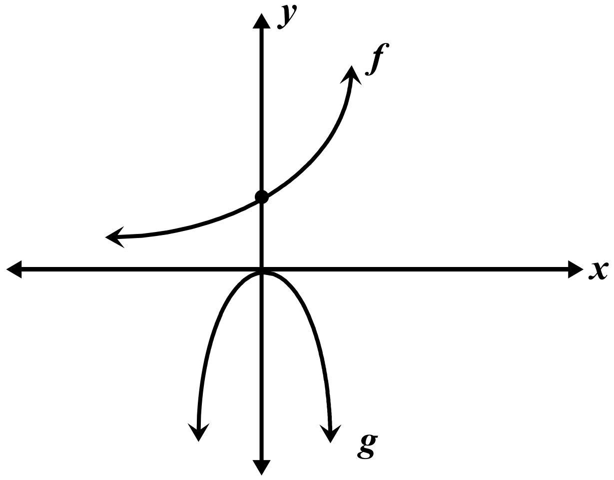
1.3 Write the inverse of *g* in the form $g^{-1}(x)=$........

1.4 Sketch the graph of g^{-1}.

1.5 Determine graphically the values of *x* for which $\log_{\frac{1}{2}} x<0$

Question 2

Sketched below are the graphs of $f(x)=3^x$ and $g(x)=-x^2$



2.1 Write down the equation of the inverse of the graph of $f(x)=3^x$ in the form $f^{-1}(x)=$.....

2.2 On a set of axes, draw the graph of the inverse of $f(x) = 3^x$

2.3 Write down the domain of the graph of $f^{-1}(x)$

2.4 Explain why the inverse of the graph of $g(x) = -x^2$ is not a function.

2.5 Consider the graph of $g(x) = -x^2$

2.5.1 Write down a possible restriction for the domain of $g(x) = -x^2$ so that the inverse of the graph of *g* will now be a function.

2.5.2 Hence draw the graph of the inverse function in

Question 3

Two functions are defined by $f(x) = (x - 4)(x + 2)$ and $g(x) = 2x - 12$.

3.1 Write down the gradient of g

3.2 Determine the co-ordinates of the turning point of *f*

3.3 Determine the range of *f*.

3.4 Determine the equation of the graph h which is the reflection of f about the $y-$ axis.

3.5 Determine the equation of the graph *k* which is the reflection of f about the $x-$ axis.

3.6 Determine g^{-1}, the inverse of *g*, in the form $y = \ldots.$

Question 4

Given: $g(x) = \left(\frac{1}{2}\right)^x$

4.1 Write the inverse of *g* in the form $g^{-1}(x) = \ldots..$

4.2 Sketch the graph of g^{-1}

4.3 Determine graphically the values of *x* for which $\log_{\frac{1}{2}} x < 0$

Question 5

The equation of graphs $f(x) = 2^x$ are given for $-3 \leq x \leq 3$.

5.1 Write down the range of f .

5.2 Write down the equation of f^{-1}, the inverse of *f*.

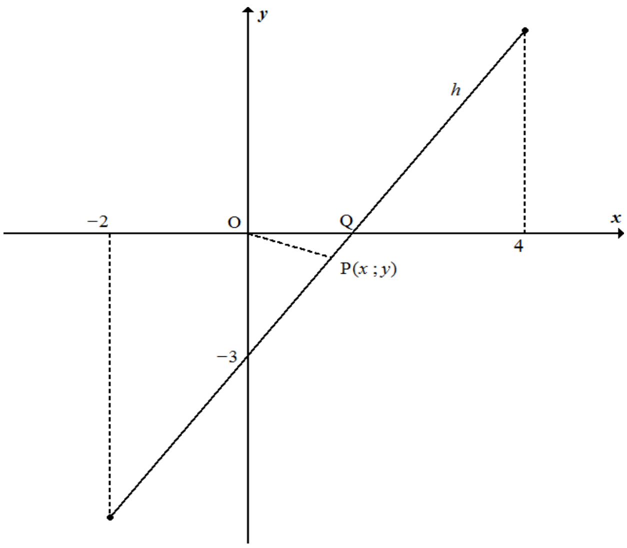
5.3 Write down the domain and the range of f^{-1} .

5.4 Draw *f* and f^{-1} on the same set of axes, showing intercepts with axes and The line(s) of symmetry.

5.5 Is f^{-1} a function or not? Give reason for your answer.

Question 6

6.1 Given: $h(x) = 2x - 3$ for $-2 \leq x \leq 4$. The x-intercept of *h is* Q.



6.1 Determine the coordinates of Q.

6.2 Write down the domain of h^{-1}.

6.3 Sketch the graph of h^{-1} in your ANSWER BOOK, clearly indicating the *y*- the intercept and the end points.

6.4 For which value(s) of x will $h(x) = h^{-1}(x)$?

6.5 P(*x* ; *y*) is the point on the graph of h that is closest to the origin. Calculate the distance OP.