S6 NUMERICAL METHODS

ESTIMATION OF ROOTS OF EQUATIONS

The root of the equation $f(x) = 0$ can be estimated by any of the following methods

(i) Analytical method (ii) Graphical method and (iii) Newton Raphson's method.

ANALYTICAL METHOD

We can deduce that the root of the equation $f(x) = 0$ lies between two values $x = x_1$ and $x = x_2$, if there is a difference between the signs to the values of the function $y = f(x)$ at $x = x_1$ and at $x = x_2$.

Note that this technique does not locate the exact root. We can then use linear interpolation to find the estimate of the root to a given accuracy.

Example

Show that the root of the equation $2x^2 + 3x - 3 = 0$ *lies between 0 and 1.* Solution

Let $f(x) = 2x^2 + 3x - 3$.

Determine $f(0) = 2(0)^2 + 3(0) - 3 = -3$ and $f(1) = 2(1)^2 + 3(1) - 3 = 2$.

Clearly, $f(0) = -3 < 0$ and $f(1) = 2 > 0$, hence there is a root between 0 and 1.

We can now use linear interpolation, say twice to find the root correct to two decimal places.

0	x_1	1
-3	0	2

Now ; $\frac{x_1 - 0}{0 - -3} = \frac{1 - 0}{2 - -3}$; $x_1 = 0.6$

Find $f(0.6) = 2(0.6)^2 + 3(0.6) - 3 = -0.48$ before obtaining the second estimate.

0.6	x_2	1
-0.48	0	2

$\frac{x_2 - 0.6}{0 - -0.48} = \frac{1 - 0.6}{2 - -0.48}$; $x_2 = 0.6774$

Hence the root of the equation is 0.68 correct to 2 decimal places.

Exercise

In each of the following cases, show that the root to the equation lies between the two given values. Hence use linear interpolation twice to determine the root of the equation correct to two decimal places.

(a) $x^3 - 3x + 1 = 0$, 1 and 2

(b) $x^3 - 5x^2 - 4 = 0$, 5 and 6

(c) $e^x + x - 4 = 0$, 0 and 2

(d) $e^x + x^3 - 4x = 0$, 1 and 1.6

(e) $e^{-x} - x = 0$, 0.5 and 1

(f) $5e^x - 4x = 6$, -2 and -1

(g) $2\ln x + 2x - 1 = 0$, 0.5 and 1.0

(h) $\ln(x + 2) + 2x + 1 = 0$, -1 and 0

(i) $\tan x - x - 1 = 0$, 1 and 1.5

(j) $2\sin x - x = 0$, 1 and 2

(k) $x = -3\tan x$, 2 and 3

(l) $2x - \tan x = 0$, 1 and 1.4

N.B: ***For trigonometric equations/ functions, set the calculator in the radian mode.***

GRAPHICAL METHOD

The root to the equation $f(x) = 0$ can also be estimated graphically by (i) obtaining coordinate points to plot (ii) drawing a suitable graph of the function $y = f(x)$ on coordinate axes.

Note, ensure that you use uniform scales on the axes and name each of the graph(s) you draw.

The x- value where the graph crosses the x axis will give the root.

Example:

Use the graphical method to show that the equation $2x^2 + 3x - 3 = 0$ *has a root between 0 and 1.*

Solution

Let $y = 2x^2 + 3x - 3$

Sub-divide the interval into smaller steps.

x	0	0.2	0.4	0.6	0.8	1
y	-3	-2.3	-1.5	-0.5	0.7	2

Note: The y-values can be determined to **one** or **two** decimal place(s), for ease of plotting.

When a graph is plotted, the curve cuts the x- axis between $x = 0.6$ and $x = 0.8$, and that is where the root is (you represent this with an arrow to show that root)

Exercise

Use the graphical method in each of the following cases to estimate the root to the equations below, in the following intervals.

(a) $x^3 - 3x + 1 = 0$, 1 and 2

(b) $x^3 - 5x^2 - 4 = 0$, 5 and 6

(c) $e^x + x - 4 = 0$, 0 and 2

(d) $e^x + x^3 - 4x = 0$, 1 and 1.6

(e) $e^{-x} - x = 0$, 0.5 and 1

(f) $5e^x = 4x + 6$, -2 and -1

(g) $2\ln x + 2x - 1 = 0$, 0.5 and 1.0

(h) $\ln(x + 2) + 2x + 1 = 0$, -1 and 0

(i) $\tan x - x - 1 = 0$, 1 and 1.5

(j) $2\sin x - x = 0$, 1 and 2

(k) $x = -3\tan x$, 2 and 3

(l) $2x - \tan x = 0$, 1 and 1.4

NEWTON RAPHSON'S METHOD (NRM)

With this technique, we start with an initial estimate and by repeatedly computing the root using an appropriate formula (i.e. by iteration), the root can be obtained to a required accuracy.

Note: Iteration shall be done always be done until when the root of the equation converges (that is when the new root obtained is the same as the previous root)

If x_n is an initial estimation/approximation to the root of the equation $f(x) = 0$, then a better estimate to the root x_{n+1} is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$; $n = 0,1,2,...$

Example

Show that the simplest formula based on Newton Raphson's method for solving the equation $2x^2 - 3x - 3 = 0$ *is* $x_{n+1} = \frac{2x_n^2 + 3}{4x_n - 3}$. *Hence taking the initial approximation to the root as* $x_0 = 0.5$, *find the root correct to three decimal places.*

Solution

Let $f(x) = 2x^2 + 3x - 3$; $f'(x) = 4x + 3$(should be obtained before obtaining $f(x_n)$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; ; \; n = 0,1,2,\ldots \text{ Using}$$

$$x_{n+1} = x_n - \frac{(2x_n^2 + 3x_n - 3)}{4x_n + 3} = \frac{4x_n^2 + 3x_n - 2x_n^2 - 3x_n + 3}{4x_n + 3} = \frac{2x_n^2 + 3}{4x_n + 3}.$$

$x_0 = 0.5$; $\quad x_1 = \dfrac{2(0.5)^2 + 3}{4\times 0.5 + 3} = 0.7000$; $\quad x_2 = \dfrac{2(0.7000)^2 + 3}{4\times 0.7000 + 3} = 0.6862$

$x_3 = \dfrac{2(0.6862)^2 + 3}{4\times 0.6862 + 3} = 0.6861$

Now $|x_3 - x_2| = |0.6861 - 0.6862| = 0.0001 < 0.0005$, the maximum possible error when the value is rounded off to three decimal places.

Therefore the root is 0.686.

Note that in the intermediate steps before the root converges, the root must be rounded off to more decimals than that of the required accuracy. The substitution of the value in the formula should also be seen at each iteration.

Example

Deduce that the simplest formula based on Newton Raphson's method for solving the equation $e^x - 4\sin x = 0$. *Hence taking the initial estimate to the root,* $x_0 = 1.2$, *find the root correct to three decimal places.*

Solution

$f(x) = e^x - 4\sin x$, $\quad f'(x) = e^x - 4\cos x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; ; \; n = 0,1,2,\ldots, \text{ Using}$$

$$x_{n+1} = x_n - \frac{(e^{x_n} - 4\sin x_n)}{e^{x_n} - 4\cos x_n} = \frac{x_n e^{x_n} - 4x_n\cos x_n - e^{x_n} + 4\sin x_n}{e^{x_n} - 4\cos x_n}$$

$x_0 = 1.2$

$$x_1 = \frac{(1.2 - 1)e^{1.2} - 4(1.2\cos 1.2 - \sin 1.2)}{e^{1.2} - 4\cos 1.2} = 1.4181$$

$$x_2 = \frac{(1.4181 - 1)e^{1.4181} - 4(1.4181\cos 1.4181 - \sin 1.4181)}{e^{1.4181} - 4\cos 1.4181} = 1.3682$$

$$x_3 = \frac{(1.3682 - 1)e^{1.3682} - 4(1.3682\cos 1.3682 - \sin 1.3682)}{e^{1.3682} - 4\cos 1.3682} = 1.3650$$

$$x_4 = \frac{(1.3650 - 1)e^{1.3650} - 4(1.3650\cos 1.3650 - \sin 1.3650)}{e^{1.3650} - 4\cos 1.3650} = 1.3650$$

Now $|x_4 - x_3| = |1.3650 - 1.3650| = 0.0000$

Therefore the root is 1.365 correct to three decimal places.

Exercise

1. Show that the iterative formula based on Newton Raphson's method for solving the equation $2x^2 - 6x + 3 = 0$ is $x_{n+1} = \dfrac{2x_n^2 - 3}{4x_n - 6}$. Hence taking the initial approximation to the root, $x_0 = 3.2$, find the root correct to two decimal places.

2. Deduce that the simplest formula based on Newton Raphson's method for solving the equation $2\sin x - x = 0$ is $x_{n+1} = \dfrac{2(x_n\cos x_n - \sin x_n)}{2\cos x_n - 1}$. Taking, $x_0 = 1.4$, find the root correct to three decimal places.

3. Prove that the simplest formula based on Newton Raphson's method for solving the equation $3xe^x - 1 = 0$ is $x_{n+1} = \dfrac{3x_n^2 e^{x_n} + 1}{3e^{x_n}(x_n + 1)}$. Hence taking, $x_0 = 0.25$, calculate the root correct to two decimal places.

4. In each of the following cases, derive the simplest formula based on Newton Raphson's method for solving the given equations. Hence taking the given initial approximation, find the root correct to three decimal places.

(a) $2x^2 \square 4 = 3x$, $x_0 = 0.5$ (b) $e^x = 10 - x$, $x_0 = 2.1$

(c) $2\ln x \square 2x \square 1 = 0$, $x_0 = 0.6$ (d) $x^3 \square 5x \square 40 = 0$, $x_0 = 3.4$

(e) $\dfrac{x}{3} + \tan x = 0$, $x_0 = 2.5$ (f) $\tan x \square x \square 1 = 0$, $x_0 = 1.3$

5. Using the Newton's iterative method, deduce that the fifth root of a number $\boldsymbol{A}$ is given by $x_{n+1} = \dfrac{1}{5}\left(4x_n + \dfrac{A}{x_n^4}\right)$. Hence taking the initial approximation as $x_0 = 2.0$, find the fifth root of $\sqrt[5]{45}$ correct to three decimal places.

Hint: Let $x = \sqrt[5]{A} \Rightarrow x^5 = A \Rightarrow x^5 - A = 0$; then proceed.

ITERATIVE FORMULAE

6. Show that the iterative formula for solving the equation $x^2 - 5x + 2 = 0$ can be written as either $x_{n+1} = 5 - \dfrac{2}{x_n}$ or $x_{n+1} = \dfrac{x_n^2 + 2}{5}$. Hence starting with $x_0 = 4$, deduce the most suitable formula for solving the above equation. Find the root correct to two decimal places.

7. Show that the iterative formula for solving the equation $x^3 - x - 1 = 0$ is $x_{n+1} = \sqrt{1 + \frac{1}{x_n}}$. Starting with $x_0 = 1$, find the root correct to 3 significant figures.

8. An iterative formula for solving the equation $f(x) = 0$ is given by $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{12}{x_n^2}\right)$.

   (a) Taking $x_0 = 2.2$, calculate the root of the equation correct to three decimal places.

   (b) Deduce the equation which is being solved by the above iterative formula.

9. (a) Show that the equation $e^x + x = 15$ has a root between 2 and 3.

   (b) Use Newton Raphson's method to find the root to the equation in (a) above, correct to three decimal places.

Hint: For such a question the initial approximation is the average of the two values 2 and 3.