CONTINUOUS PROBABILITY DISTRIBUTION

This is a distribution which takes on any value within a given interval.

Summary:

A probability density function **(p.d.f)** is a function that defines the probability of an event to occur.

A continuous p.d.f ***f(x)*** defined over the interval $a \le x \le b$ is such that**:**

(a) The total area under ***f(x)*** = sum of all probabilities = **1.**

$$\int_a^b f(x)\,dx = 1.$$

(b) $$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx.$$

NOTE:

(i) The values of $P(x_1 < X < x_2)$, $P(x_1 \le X < x_2)$, $P(x_1 < X \le x_2)$ and $P(x_1 \le X \le x_2)$ are the same.

(ii) The values $P(X = x_1) = P(X = x_2) = 0$ since **X** deals with a range of values

(c) Expectation**,** $$E(X) = \int_a^b x f(x)\,dx.$$

(d) Variance, $\mathbf{Var(X) = E(X^2) - E^2(X)}$.

where $\mathbf{E(X^2)} = \int_a^b x^2 f(x)\,\mathbf{dx}$, $\quad \mathbf{E^2(X) = [E(X)]^2}$.

(e) Standard deviation $\boldsymbol{\sigma} = \sqrt{\textbf{variance}}$

(f) For a continuous r.v **X** and constants **a** and **b,**

(i) E(a) = a	**Var(a) = 0**
(ii) E(aX) = aE(X)	**Var(aX) = a^{2}Var(X)**
(iii) E(aX + b) = aE(X) + b	**Var(aX + b) = a^{2}Var(X)**

(g) Median is the value **m** which satisfies the relation $\int_a^m f(x)\,\mathbf{dx} = \mathbf{0.5}$.

I.e The median encloses an area of **0.5** below it.

(h) Lower quartile is the value $\mathbf{q_1}$ which satisfies the relation $\int_a^{q_1} f(x)\,\mathbf{dx} = \mathbf{0.25}$.

I.e The lower quartile encloses an area of **0.25** below it.

(I) Upper quartile is the value $\mathbf{q_3}$ which satisfies the relation $\int_a^{q_3} f(x)\,\mathbf{dx} = \mathbf{0.75}$.

I.e The upper quartile encloses an area of **0.75** below it.

(j) Interquartile range = $\mathbf{Q_3 - Q_1}$

p

(k) The J^{th} percentile is the value **p** that satisfies the relation $\int_a^p f(x)\,dx = \frac{J}{100}$.

NOTE: The median, quartiles and percentiles of a p.d.f defined over different intervals are obtained by first investigating the interval in which they are located.

(L) The graph of the p.d.f $f(x)$ can either be linear or a curve.

(m) Mode is value of **x** at which the p.d.f $f(x)$ attains its maximum value.

The graph of $f(x)$ gives the location of the mode. If the p.d.f $f(x)$ is non linear, the value **x** at which $f(x)$ has a maximum value occurs when $f^{1}(x) = 0$

provided $f^{11}(x) < 0$.

EXAMPLES:

1. The p.d.f of a continuous **r.v X** is given by**:**

$$f(x) = \begin{cases} \beta x & , \ 0 \le x \le 1 \\ \frac{1}{2}\beta(3 - x) & , \ 1 \le x \le 3 \\ 0 & , \ \text{otherwise} \end{cases}$$

Find**:**

(i) the value of □

(ii) **P(X □ 2)**

(iii) **P(X □ 2)**

(iv) **P(X □ 1□4)**

(v) **P(0□8 □ X □ 2)**

(vi) **P(X □ 1| □ 0□6|)**

(vii) **P(|X □ 1| □ 0□6)**

(viii) **P(0□2 □ X □ 2□5/X □ 0□7)**

(ix) the mode, mean and standard deviation of **X.**

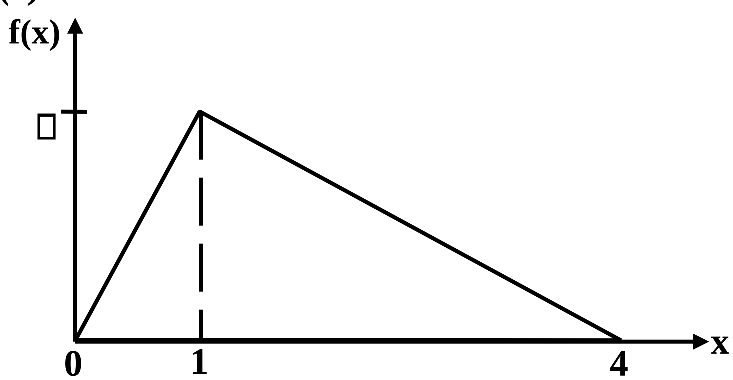
(x) **E(3X + 5)**

(xi) **Var(3X + 5)**

(xii) the median and semi-interquartile range of **X.** **(xiii)** the value of **b** such that **P(X b) = 0 6.□ □**

(xiv) the **30th** to **80th** percentile range of **X.**

2. The p□d□f ***f(x)*** of a **r□v X** takes on the form shown in the sketch below**:**



Find the**:**

(i) value of □

(ii) equations of the p.d.f

(iii) **$P(0.5 < X \le 2)$**

(iv) mean of **X.**

(v) median of **X.**

3. The p.d.f of a continuous **r.v X** is such that**:**

$$f(x) = \begin{cases} \beta x(6 - x)^2, & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$$

Find the**:**

(i) value of β

(ii) mode of **X**

(iii) mean of **X**

4. The outputs of **9** machines in a factory are independent random variables each with probability density function given by

$$f(x) = \begin{cases} \beta x, & 0 \le x \le 10 \\ \beta(20 - x), & 10 \le x \le 20 \\ 0, & \text{otherwise} \end{cases}$$

Find the**:**

(i) value of β**.**

(ii) expected value and variance of the output of each machine.

Hence or otherwise find the expected value and variance of the total output from all machines.

5. The mass **X kg** of maize flour produced per hour is modeled by a continuous r.v whose p.d.f is given by**:**

$$f(x) = \begin{cases} \beta x^2 & , \quad 0 \le x \le 2 \\ \beta(6 - x) & , \quad 2 \le x \le 6 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch the p.d.f of **X.** Hence state the mode of **X**

(b) Find the**:**

(i) value of β

(ii) $P(|X - 3| < 2)$

(iii) mean mass produced per hour

(c) Given that maize flour is sold at **sh 2400** per kg and the cost of running the production is **sh 200** per hour, taking **shs Y** as the profit made per hour.

(i) Express **Y** in terms of **X.**

(ii) Find the expected value of **Y.**

6. A r.v **X** has the following p.d.f

$$f(x) = \begin{cases} \beta x & , \ 1 \le x \le 3 \\ \lambda(4-x) & , \ 3 \le x \le 4 \\ 0 & , \ \text{otherwise} \end{cases}$$

(a) Show that $\lambda = 3\beta$.

(b) Find **:**

(i) the values of β and λ

(ii) the mean and variance of **X**

(iii) the median of **X**

(iv) $P(3 \le X \le 4 / X \ge 2)$

7. A r.v **X** has the following p.d.f.

$$f(x) = \begin{cases} \frac{2}{13}(x + 1) & , \ 0 \le x \le a \\ \frac{2}{13}(5 - x) & , \ a \le x \le b \\ 0 & , \ \text{otherwise} \end{cases}$$

Find the**:**

(i) values of **a** and **b**.

(ii) median of **X.**

(iii) **P(X 0 5 0 25 X 1)**

CUMULATIVE DISTRIBUTION FUNCTION *F(x)*

This function gives the accumulated probability up to **x.** It is obtained by integrating the p.d.f as follows:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt.$$

The cumulative distribution function is sometimes known as a distribution function

PROPERTIES OF *F(x)*

(i) **F(x)** must be defined over the interval $-\infty < x < \infty$.

(ii) $0 \le F(x) \le 1$, for all values of **x.**

(iii) $P(x_1 \le X \le x_2) = P(X \le x_2) - P(X \le x_1) = F(x_2) - F(x_1)$

(iv) The median, **m,** lower quartile, q_1, and upper quartile q_3 are the values for which $F(m) = \frac{1}{2}$, $F(q_1) = \frac{1}{4}$ and $F(q_3) = \frac{3}{4}$ respectively.

(v) $P(X \le x) + P(X > x) = 1$

$\Rightarrow$ The complementary cumulative distribution function

$$P(X > x) = 1 - P(X \le x) = 1 - F(x)$$

(vi) The p.d.f *f(x)* can be obtained by differentiating the cumulative distribution

$\Rightarrow \quad F^1(x) =$ p.d.f $f(x)$

EXAMPLES:

1. The p.d.f of a continuous **r.v X** is given by:

$$f(x) = \begin{cases} \frac{1}{2}\beta(x-1), & 1 \le x \le 3 \\ \frac{1}{2}\beta(7-x), & 3 \le x \le 7 \\ 0, & \text{otherwise} \end{cases}$$

Find**:**

(i) the value of k, hence $f(x)$

(ii) the cumulative distribution function **F(x)** and sketch it.

(iii) $P(2.8 \le X \le 5.2)$

(iv) $P(X \ge 4)$

(v) the median of **X.**

(vi) the interquartile range of X

(vii) the **20th** percentile of **X.**

Solution:

(ii) Note: **F(x)** is concave up parabola over the interval $1 \le x \le 3$ and concave down parabola over the interval $3 \le x \le 7$.

2. A continuous r.v **X** has the following p.d.f.

$$f(x) = \begin{cases} k(3 - x) & , \quad 1 \le x \le 2 \\ k & , \quad 2 \le x \le 3 \\ k(x - 2) & , \quad 3 \le x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch $f(x)$, hence deduce the mean and median of **X.**

(b) Find**:**

(i) the value of k.

(ii) the cumulative distribution function **F(x)** and sketch it.

(iii) $P(X > 3.5 / 3 < X < 4)$

3. The distribution function of a continuous r.v X is as follows:

$$P(X \le x) = \begin{cases} 0 & , \quad x < 1 \\ \frac{1}{12}(x-1)^2 & , \quad 1 \le x < 3 \\ \frac{1}{24}(\beta x + \lambda - x^2) & , \quad 3 \le x < 7 \\ 1 & , \quad x \ge 7 \end{cases}$$

Find:

(i) the values of β and λ

(ii) $P(X > 4)$

(iii) the median of X

(iii) the p.d.f of X

(iv) the mean, μ of the distribution

(v) $P(|X - \mu| < 0.8)$

4. The cumulative distribution of a continuous r.v X is such that:

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ 2x - 2x^2 & , \quad 0 \le x < 0.25 \\ \text{[illegible]}\ x & , \quad 0.25 \le x < 0.5 \end{cases}$$

$$\square\square\ \square\ \square\ 2x^2\ \square\ x \quad , \quad 0\square 5\ \square\ x\ \square$$
$$0\square 75\ \square\square\square \quad 1 \quad , \quad x\ \square$$
$$0\square 75$$

Find**:**

(i) the values of □ and □ **,** hence sketch **F(x).**

(ii) P($|$ **X □ 0□375**$|$**□ 0□25)**

(iii) the p□d□f of **X** and sketch it**,** hence deduce the mean and median of **X.**

5. A continuous r□v **X** is distributed as follows**:**

$$\mathbf{P(\ X >)\quad x\ \square\ a + bx^3 \quad , \quad 0 \quad x \quad \square\ \square 4}$$

(i) By first finding the cumulative distribution of **X** or otherwise, find the values of **a** and **b.**

(ii) Show that **E(X) = 3,** and find the standard deviation □ of **X.**

UNIFORM DISTRIBUTION

This distribution is sometimes called a rectangular distribution.

Summary:

If a **r□v X** is uniformly distributed over the interval □**a, b**□**,** then**:**

$$\square\square\ \mathbf{b\ \square\frac{1}{\ }\ a \quad , \quad a\ \square\ x\ \square\ b}$$
$$\square\square\quad \mathbf{0 \quad , \quad otherwise}$$

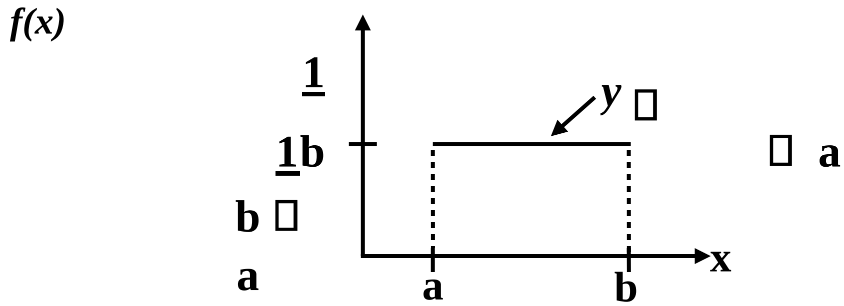
(i) the p□d□f of **X** is given by**:** $f(x)$ □ □

(ii) the mean of **X** is $\frac{a+b}{2}$.

(iii) the variance of **X** is $\frac{(b-a)^2}{12}$.

(iv) the median of **X** is $\frac{a+b}{2}$.

(v) the graph of *f(x)* is as follows:



EXAMPLES:

1. A **r.v X** is uniformly distributed over the interval **[a, b].**

(a) State the p.d.f of **X** and sketch it.

(b) Show that:

(i) the mean of **X** is $\frac{a+b}{2}$.

(ii) the variance of **X** is $\frac{(b-a)^2}{12}$.

(iii) the median of **X** is $\frac{a+b}{2}$.

(c) Find the cumulative distribution function of **X** and sketch it.

2. (a) A **r.v X** is uniformly distributed over the interval $[2, 5]$. Find

$$P(X < 2.3 / X < 4.5)$$

(b) The number of vehicles crossing a roundabout take on a **r.v X** with uniformly distribution over the interval $[x_1, x_2]$. If the expected number of vehicles crossing the roundabout is **15** with variance **3,** calculate the**:**

(i) values of x_1 and x_2.

(ii) probability that at least **14** vehicles cross the roundabout**.**

(iii) probability that the number of vehicles crossing the roundabout lies within one standard deviation of the mean**.**

EER:

1. The p.d.f of a continuous r.v **X** is given by

$$f(x) = \begin{cases} k(1-x^2) & , \quad 0 < x < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find:

(i) the value of □.

(ii) the mean, **μ** and standard deviation, □ of **X.**

(iii) **E(8X + 3)** and **Var(8X + 3)**

(iv) $P(|X - \mu| □ \sigma)$

[Ans: (i) 1.5 (ii) 0.375, 0.2437 (iii) 6, 3.8 (iii) 0.6145]

2. The p.d.f of a continuous r.v **X** is given by

$$f(x) = \begin{cases} \frac{2}{3}x & , \quad 0 □ x □ 1 \\ \frac{1}{3}(3 - x) & , \quad 1 □ x □ 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Show that ***f(x)*** represents a probability density function**.**

(b) Find the**:**

(i) median of **X**

(ii) **80th** percentile of **X.**

(iii) value of **b** such that **P(X □ b) □ 0.6.**

(iv) expressions for **P(X □ x)** and sketch it.

[Ans: b(i) 1.2679   (ii) 1.9046   (iii) 1.4508]

3. The p.d.f of a continuous r.v **X** is given by

$$f(x) = \begin{cases} \alpha x - \beta x^2 , & 0 \le x \le 2 \\ 0 , & \text{otherwise} \end{cases}$$

Given that the mean of **X** is **1,** find the**:**

(i) values of α and β**.**

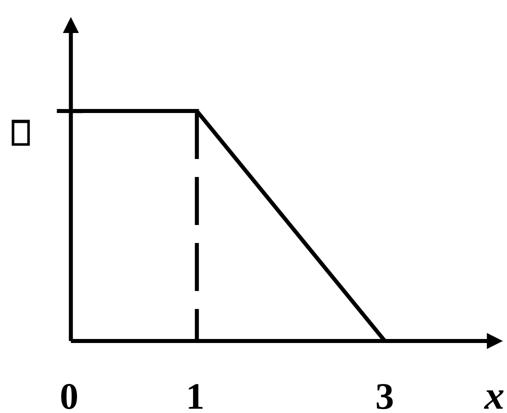
(ii) variance of **X**

(iii) mode of **X**

[Ans: (i) 1.5 , 0.75   (ii) 0.2 (iii) 1]

4. The p.d.f ***f(x)*** of a **r.v X** takes on the form shown in the sketch below**:**

f(x)



Find**:**

(i) the value of k

(ii) the equations of the p.d.f **(iii)** the mean of **X.**

(iv) $P(|X - 1.25| < 0.75)$

(v) $P(0.2 < X < 1.5 \mid X > 0.6)$

[Ans: (i) $\frac{1}{2}$ (iii) 1.0833 (iv) 0.625 (v) 0.5982]

5. A r.v **X** is uniformly distributed with mean **7.5** and variance **0.75** over the interval **[a, b]**. Find:

(i) the values of **a** and **b.**

(ii) the p.d.f of the distribution.

(iii) $P(7.2 < X < 8.4)$

(iv) **80**th percentile of **X.**

(v) probability that **X** lies within one standard deviation of the mean.

(vi) cumulative distribution function of **X.**

[Ans: (i) 6, 9 (iii) 0.4 (iv) 8.4 (v) 0.5774]

6. The time taken to perform a particular task **t hours** is given by the p.d.f:

$$f(t) = \begin{cases} 10\beta t^2 & , \quad 0 \le t \le 0.6 \\ 9\beta(1 - t) & , \quad 0.6 \le t \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Find the:

(i) value of β

(ii) most likely time.

(iii) expected time.

(b) Determine the probability that the time will be:

(i) more than **48** minutes.

(ii) between **24**and **48** minutes.

[Ans: (a) (i) (ii) $\frac{25}{36}$ 0.6 (iii) 0.591 (b) (i) 0.125 (ii) 0.727]

7. A continuous r.v **X** has the following p.d.f

$$f(x) = \begin{cases} \square\left(x \square k^{\frac{1}{\square}}\right)^{\square} & , \quad 0 \le x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Given that **P(X > 1) = 0.8,** find the:

(i) values of ***k*** and □.

(ii) probability that **X** lies between **0.5** and **2.5**

(iii) mean of **X**

[Ans: (i) □1, $^2/_{15}$ (ii) 0.6667 (iii) 1.8]

8. A continuous **r.v X** has the following p.d.f **:**

$$f(x) = \begin{cases} \beta x^2 & , \quad 0 \le x \le 2 \\ \beta(6 - x) & , \quad 2 \le x \le 6 \end{cases}$$

$$0 \quad , \quad \text{otherwise}$$

(a) Sketch ***f(x)***

(b) Find**:**

(i) the value of □.

(ii) the median of **X.**

(iii) P(|X 2 75 | 1 25)□ □□ □

□Ans: b(i) $\frac{3}{32}$ (ii) 2□734 (iii) 0□7070 □

9. A continuous r□v **X** has the following p□d□f

$$f(x) = \begin{cases} k & , \quad 0 \le x \le 2 \\ k(2x - 3) & , \quad 2 \le x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x)***

(b) Find**:**

(i) the value of **k.**

(ii) the mean and variance of **X.**

(iii) P(X □ 2□5)

(iv) P(1 □ X □ 2□5)

(v) P(0 □ X □ 2/X □ 1)

□Ans: (i) (ii), $\frac{1}{4}$ $\frac{43}{24}$ 0□8316 (iii) 0□3125 (iv) 0□4375 $\frac{1}{3}$ (v) □

10. The cumulative distribution of a continuous **r.v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x < 1 \\ \alpha(x-1)^2 & , \quad 1 \le x \le 3 \\ \beta(14x - x^2 - 25) & , \quad 3 \le x \le 7 \\ 1 & , \quad x > 7 \end{cases}$$

Find**:**

(i) the values of α and β**.**

(ii) $P(2.8 \le X \le 5.2)$

(iii) the median of **X**

(iv) the p.d.f of **X** and sketch it**.**

(v) the mean, **μ** of **X.**

(vi) $P(|X - \mu| \le 0.8)$

[Ans: $\frac{1}{12}$ $\frac{1}{24}$ **(i), (ii) 0.595 (iii)** $\frac{11}{3}$ **3.45 (v) (vi) 0.5578]**

11. The number of boats **X** crossing a river is uniformly distributed between **150** and **210.**

(a) State the p.d.f of the distribution**.**

(b) Find the**:**

(i) probability that between **170** and **194** boats cross the river**.**

(ii) expected number of boats to cross the river**.**

(iii) standard deviation for the distribution**.**

□Ans: b(i) 0□4 (ii) 180 (iii) 7□7460 □

12. The cumulative distribution of a continuous **r□v X** is such that**:**

□ 0 , x □ 1 □□□

^{2} □ 2x □ 1) , 1 □ x □ 2 0□5(x

F(x) □ □

□ βx □ λx^2 □3□5 , 2 □ x □3 □□ 1

, x □ 3

□

Find**:**

(i) the values of □ and □**.** Hence sketch **F(x)**

(ii) P(1□5 □ X □ 2□5/X □ 2)

(iii) the p□d□f of **X** and sketch it**.** Hence deduce the mean, mode and median

□Ans: (i) 3 , □0□5 (ii) 0□75 (iii) 2, 2, 2 □

13. A continuous r□v **X** has the following p□d□f.

□ *k*(3 □ x) , 1 □ x □ 2

□

$$f(x) \;□\; □□ \quad k \qquad , \quad 2$$
$$□\, x \;\; □\, 3$$

$$□\;\; k(x \;□\; 2) \qquad , \quad 3 \;□\, x \;\; □4$$
$$□□ \qquad 0 \qquad\qquad , \quad \text{otherwise}$$

(a) Sketch ***f(x),*** hence deduce the mean and median of **X.**

(b) Find**:**

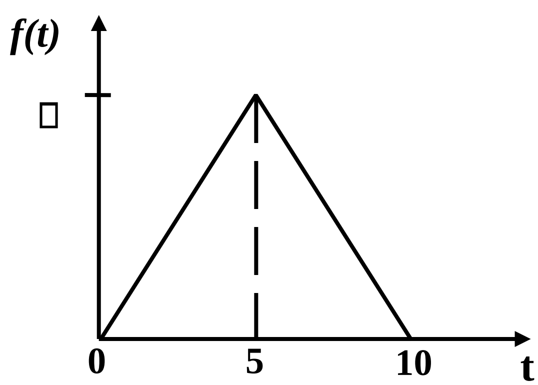
(i) the value of ***k*.**

(ii) **P(X □ 3□5/ 3 □ X □ 4)**

(iii) the **80th** percentile of **X.**

□Ans: (a) 2□5, 2□5 b(i) 0□25 (ii) 0□583 (iii) 3□5492 □

14. The time**, T,** taken to complete a certain task can be modeled as in the diagram below, where **t** is the time in minutes.



Determine the:

(i) value of □

(ii) equations of the p□d□f

(iii) **E(T)**

(iv) probability that the task will be completed between **4** and **7** minutes.

(v) probability that the task will be completed in less than **2** minutes

□Ans: (i) 0□2 (iii) 5 (iv) 0□5 (v) 0□08 □

15. A r□v **X** is uniformly distributed with variance **6□75** over the interval

3 x b.□ □ Find**:**

(i) the value of **b.**

(ii) the p□d□f of the distribution**.** **(iii) P(5 □ X □ 9**/**X □** 7**)**

□Ans: (i) 12 (iii) 0□4 □

16. A **r□v X** is uniformly distributed over the interval **□a, b□.**

(a) State the p□d□f of **X** and sketch it**.**

(b) Show that**:**

(i) the mean of **X** is $\frac{a \square b}{2}$.

(ii) the variance of **X** is $\frac{(b \square a)^2}{12}$.

(iii) the median of **X** is $\frac{a \square b}{2}$.

(c) Find the cumulative distribution function of **X** and sketch it.

17. A r.v **X** has the following cumulative distribution function

$$F(x) = \begin{cases} 0 & , \quad x < a \\ \dfrac{x - a}{b - a} & , \quad a \le x \le b \\ 1 & , \quad x > b \end{cases}$$

(a) Sketch **F(x)**

(b) Find the p.d.f of **X.** Hence show that the variance of **X** is $\dfrac{(b - a)^2}{12}$.

18. A **r.v X** is uniformly distributed with lower quartile **5** and upper quartile **9** in the interval **[a, b].** Find the**:**

(i) values of **a** and **b.**

(ii) $P(6 < X < 7)$

(iii) probability that **X** lies within one standard deviation of the mean**.**

(iv) cumulative distribution function of **X.**

[Ans: (i) 3, 11 (ii) 0.125 (iii) 0.5774]

19. The p.d.f of a r.v **X** is given by**:**

$$f(x) = \begin{cases} \dfrac{1}{5} & , \quad 0 \le x \le 5 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) Identify the distribution

(ii) Find $\mathbf{P(|X - \mu| < \sigma)}$ where μ and σ is the mean and standard deviation of **X** respectively.

[Ans: (ii) 0·5774]

20. The life time in years of a battery is known to be uniformly distributed with mean **4** and variance $\mathbf{4/3}$ **,** issued with a three years guarantee. If two such batteries are picked at random, find the probability that both will be replaced under the guarantee.

[Ans: (ii) 0·0625]

21. A r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} 3x^{a} & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the**:**

(i) value of **a.**

(ii) median of **X.**

[Ans: (i) 2 (ii) 0·7937]

22 A r.v **X** has the following cumulative distribution function

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \dfrac{1}{32}(x^4 - 8x^3 + \beta x^2) , & 0 \le x \le 4 \\ 1 & , \quad x > 4 \end{cases}$$

Find**:**

(i) the value of **β.**

(ii) P(X < 2)

(iii) the p.d.f of **X** **(iv)** the mode of the distribution

[Ans: (i) 18 (ii) 0.75 (iv) 1]

23. The cumulative distribution of a continuous **r.v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ 0.25x^2 & , \quad 0 \le x < 1 \\ \beta x + \alpha & , \quad 1 \le x < 2 \\ 0.25(5 - x)(x - 1) , & 2 \le x < 3 \\ 1 & , \quad x \ge 3 \end{cases}$$

Find**:**

(i) the values of α and β**.**

(ii) **P(3 < 2X < 5)**

(iii) the p□d□f of **X** and sketch it**.**

(iv) the mean, **μ** of **X.**

□Ans: (i) 0□5, □0 □25 (ii) 0□4375 (iv) 1□5□

24. A continuous r□v **X** is distributed as follows**:**

P(X□ x) □ □□□□□□*x*^{3} , 0 □ x □ 3

(i) Find the values of □ and □**.**

(ii) Show that **E(X) = 2□25,** and find the standard deviation □ of **X.**

□Ans: (i) 1 , $\frac{-1}{27}$ (ii) 0□581□

25. The p□d□f of r□v **X** is given by

□□ □x(16 □ x2) , 0 □ x □ 4

f(x) □ □

□□ 0 , otherwise

Find the**:**

(i) value of **β**

(ii) mode of **X**

(iii) mean of **X**

□Ans: (i) $\frac{1}{64}$ (ii) 2□3094 (iii) $\frac{32}{15}$ □

26. A **r□v X** is uniformly distributed over the interval **□2, 6□.** Find

(i) $P(X < 2.9 / X < 3.5)$

(ii) the variance of **X**

$$\left[\textbf{Ans: (i) } 0.4 \quad \textbf{(ii) } \frac{4}{3}\right]$$

27 A continuous r.v **X** has the following p.d.f.

$$f(x) = \begin{cases} 2 - 4x & , \quad 0 \le x < 0.25 \\ 1 & , \quad 0.25 \le x < 0.5 \\ 4x - 1 & , \quad 0.5 \le x < 0.75 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x),*** hence deduce the mean, **μ** and median of **X.**

(b) Find**:**

(i) the cumulative distribution function of **X.**

(ii) $P(|X - \mu| < 0.2)$

$$\left[\textbf{Ans: (a) } 0.375, \quad 0.375 \qquad \textbf{b(ii) } 0.4225\right]$$

28. The cumulative distribution of a continuous **r.v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x < 1 \\ \frac{1}{12}(x - 1)^2 & , \quad 1 \le x < 3 \\ \frac{1}{24}(14x - x^2 - 25) & , \quad 3 \le x < 7 \end{cases}$$

$1 \quad , \quad x \geq 7$ Find:

(i) $P(2.8 < X < 5.2)$

(ii) the median of **X**

(iii) the p.d.f of **X** and sketch it.

(iv) the standard deviation of **X.**

(Ans: (i) 0.595 (ii) 3.45 (iv) 1.2472)

29. The number of boats **X** crossing a river is uniformly distributed between **30** and **110** boats. Find the:

(i) probability that at least **90** boats cross the river.

(ii) expected number of boats to cross the river.

(iii) standard deviation for the number of boats to cross the river.

(iv) probability that **X** lies within one standard deviation of the mean.

(v) upper quartile for the number of boats to cross the river.

(vi) **25th** percentile for the number of boats to cross the river.

(vii) cumulative distribution function of **X** and sketch it.

(Ans: (i) 0.25 (ii) 70 (iii) 23.094 (iv) 0.5774 (v) 90 (vi) 50)

30. The p.d.f of r.v **X** is given by

$$f(x) = \begin{cases} x(3x - x^2) \quad , \quad 0 \leq x \leq 3 \end{cases}$$

□□ **0** **,** **otherwise**

Find the**:**

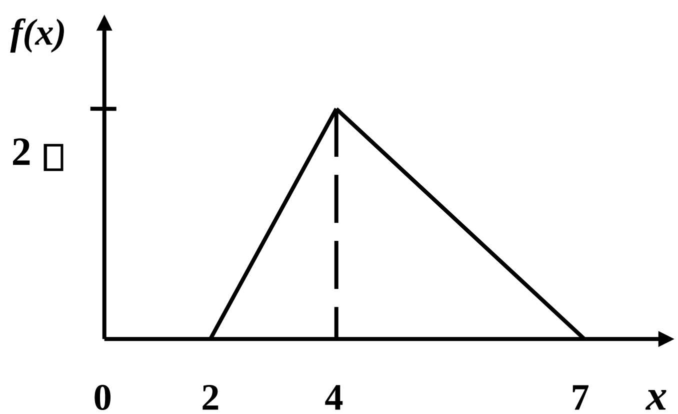
(i) value of **β**

(ii) mode of **X**

(iii) mean of **X**

□Ans: (i) $\frac{4}{27}$ (ii) 2 (iii) 1□8 □

31 The p□d□f ***f(x)*** of a **r□v X** takes on the form shown in the sketch below**:**



Find the**:**

(i) value of □

(ii) equations of the p□d□f **(iii)** median of **X.**

□Ans: (i) 0□2 (iii) 4□2614 □

32. The weekly demand for petrol **X** in thousands of units at the petrol station is given by the p□d□f

$$f(x) = \begin{cases} \beta x(\lambda - x) & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) Given that the mean weekly demand is **625** units, find the values of **β** and **λ**. Hence obtain the mode of **X.**

(ii) If every week the petrol station stocks **750** units of petrol, find the probability that in a given week the petrol station will be unable to meet the demand for petrol.

(iii) Find the amount of petrol that should be stocked in order to be **85·05%** certain that the demand for petrol in that week will be met**.**

[Ans: (i) 1·5, 2 (ii) 0·3672 (iii) 900units]

33 A r□v **X** has the following p□d□f.

□□ 3 **2** **a(x □ a)** **,** □**a** □ **x** □ **0** ***f(x)*** □ □□□ 3 **1** **a(2a**

□ **x)** **,** **0** □ **x** □ **2a** □ **0** **,** **otherwise**

□

□□

Find**:**

(i) the value of **a**.

(ii) the expressions for **P(X** □ **x)** and sketch it

(iii) the median of **X.**

(iv) P(X □ **1□5/X** □ **0)**

□**Ans: (i) 1 (iii) 0□2679 (iv) 0□9375** □

34. A r□v **X** has the following p□d□f.

□□ 3 **1** **a(x □ 2)** **,** □**a** □ **x** □ **1**

□

f(x) □ □□ **a** **1** **(2** □ **x)** **,** **1** □ **x** □ **a**

□

□ **0** **,** **otherwise**

□□

Find**:**

(i) the value of **a**.

(ii) **P(X □ 0)**

(iii) the lower quartile of **X.**

[Ans: (i) 2 (iii) $^1/_3$ (iii) □0.2679]

35 A continuous r.v **X** has the following p.d.f

$$f(x) = \begin{cases} \lambda \sin x , & 0 \le x \le \pi \\ 0 , & \text{otherwise} \end{cases}$$

Find**:**

(i) the value of λ.

(ii) $P\left(X \ge \frac{\pi}{3}\right)$

(iii) the median of **X.**

[Ans: (i) 0.5 (ii) 0.75 (iii) $\frac{\pi}{2}$]

36. A continuous r.v **X** has the following p.d.f

$$f(x) = \begin{cases} \lambda(1 - \cos x) , & 0 \le x \le \pi/2 \\ \lambda \sin x , & \pi/2 \le x \le \pi \end{cases}$$

$$\quad 0 \qquad , \qquad \text{otherwise}$$

(a) Find**:**

(i) the value of λ.

(ii) $P\left(\frac{\pi}{3} < X < \frac{3\pi}{4}\right)$

(b) Show that the mean, μ of the distribution is $1 - \frac{\pi}{4}$.

(Ans: (i) $\frac{2}{\pi}$ (ii) 0.6982)

37 A r.v **X** has the following p.d.f.

$$f(x) = \begin{cases} \lambda \cos x & , \quad 0 \le x \le \frac{\pi}{4} \\ \lambda \sin x & , \quad \frac{\pi}{4} < x \le \frac{\pi}{2} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find**:**

(i) the value of λ, hence sketch $f(x)$.

(ii) $P\left(x > \frac{\pi}{4} \,\middle/\, 0 < x < \frac{\pi}{3}\right)$

(iii) the mean of **X.**

$\left(\text{Ans: (i) } \frac{\sqrt{2}}{2} \quad \text{(ii) } 0.2265 \quad \text{(iii) } \frac{\pi}{4}\right)$

38. The cumulative distribution of a continuous **r.v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \; x < 0 \\ \beta \sin^{-1} x & , \; 0 \le x \le 1 \\ \lambda \tan^{-1} x & , \; 1 \le x \le \sqrt{3} \\ 1 & , \; x \ge \sqrt{3} \end{cases}$$

(a) Find**:**

(i) the values of β and λ**.**

(ii) $P(0.5 \le X \le 1.5)$

(iii) the p.d.f of **X.**

(b) Show that the mean, μ of the distribution is $\frac{3}{2\pi}\left(1 + \ln 2\right)$

$\left(\text{Ans: (i) } \frac{3}{\pi}, \; \frac{3}{2\pi} \quad \text{(ii) } 0.6885\right)$

39. A r□v **X** has the following cumulative distribution function.

□ 0 , x □ 1

□□□ 22□□ x)x) ,, 12 □□

x x □□ 23 β(x

F(x) □ □ □ λ(x

□

□□ 1 , x □ 3

Find**:**

(i) the values of □ and □**.**

(ii) the p□d□f of **X**

(iii) the mean**, μ** of **X**.

(iv) P(X □ μ | □ 0□5)

□Ans: (i) 1⁄12, 1⁄4 (ii) 2□0556 (iii) 0□5412 □

40. The p□d□f of a **r□v X** is given by**:**

□ λ f(x) □ □□ x^2 □

1 , 0 □ x □ 1

□□ 0 , otherwise

Show that the**:**

(i) value of **λ □ $\frac{4}{\pi}$.**

(ii) $P\left(\frac{1}{3} < X < \sqrt{3}\right) = \frac{1}{3}$.

(iii) $E(X) = \frac{2\ln 2}{\pi}$.

(iv) median of the distribution is $\tan\frac{\pi}{8}$

41. The cumulative distribution of a continuous **r.v X** is given by:

$$F(x) = \begin{cases} 0 & , x < 0 \\ \beta\tan^{-1}x & , 0 \le x \le \sqrt{3} \\ 1 & , x > \sqrt{3} \end{cases}$$

(a) Find:

(i) the value of β.

(ii) P(X > 1)

(b) Show that the:

(i) median of **X** is **tan** $\frac{\pi}{6}$.

(ii) 75th percentile of **X** is **tan** $\frac{\pi}{4}$.

(c) By stating the p.d.f of **X,** show that **E(X)** $= \frac{3\ln 2}{\pi}$.

42. The times of arrival of a bus at its stage are uniformly distributed between the interval **9:00am** to **2:00pm.** Find the**:**

(i) mean and variance of the bus's time of arrival

(ii) probability that the time of arrival does not exceed **1:00pm.**

(Ans: (i) 11.5h, $^{25}/_{12}$ (iii) 0.8)

43. A **r.v X** is uniformly distributed over the interval **[a, b].**

(a) State the p.d.f of **X** and sketch it**.**

(b) Show that the lower quartile of **X** is $\mathbf{\dfrac{3a+b}{4}}$ and the upper is $\mathbf{\dfrac{a+3b}{4}}$.

44. A r.v **X** has the following p.d.f

$$f(x) = \begin{cases} \beta x & , \quad 1 \le x \le 3 \\ \lambda(4-x) & , \quad 3 < x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x)***

(b) Find**:**

(i) the values of **β** and **λ**

(ii) the mean of **X**

(iii) $\mathbf{P(3 \le X \le 4 \mid X \ge 2)}$ **(Ans: b(i) $^{2}/_{11}$, $^{6}/_{11}$ (ii) 2.4848 (iii) 0.375)**

45. A continuous r.v **X** has the following p.d.f.

$$f(x) = \begin{cases} \frac{x}{3} - \frac{2}{3} & , \quad 2 \le x \le 3 \\ \lambda & , \quad 3 \le x \le 5 \\ 2 - \beta x & , \quad 5 \le x \le 6 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Find the values of λ and β

(b) Sketch ***f(x)***, hence deduce the mean, **μ** of **X.**

(c) Find the**:**

(i) variance of **X.**

(ii) **E(3X)** and **Var(3X)**

[Ans: (a) $\frac{1}{3}$, $\frac{1}{3}$ (b) 4 c(i) $\frac{101}{6}$ (ii) 12, 151·5]

46. The mass **X kg** of maize flour produced per hour is modeled by a continuous r.v whose p.d.f is given by**:**

$$f(x) = \begin{cases} \lambda(4 - x^2) & , \quad 0 \le x \le 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Find the**:**

(i) value of λ

(ii) mean mass produced per hour

(b) Given that maize flour is sold at **£ 8** per kg and the cost of running the production is **£ 1** per hour, find the**:**

(i) expected profit per hour**.**

(ii) probability that in an hour the profit will exceed **£ 11.**

[Ans: a(i) $\frac{3}{16}$ (ii) $\frac{3}{4}$ b(i) £ 5 (ii) 0.0859]

47. A r.v **X** is uniformly distributed over the interval $a \le X \le b$. Given that **X** is distributed with mean **9** and variance **12,** find:

(i) the values of **a** and **b.**

(ii) $P(X \le 10)$

[Ans: (i) 3, 15 (ii) $\frac{7}{12}$]

48. A **r.v X** is uniformly distributed over the interval **[a, b].**

(a) State the p.d.f of **X** and sketch it**.**

(b) Show that $P(x_1 \le X \le x_2) = \frac{x_2 - x_1}{b - a}$

49. A continuous r.v **X** has the following p.d.f

$$f(x) = \begin{cases} \lambda x(3 - x), & 0 \le x \le 2 \\ \lambda(4 - x), & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Find**:**

(i) the value of λ**.**

(ii) the mean of **X.**

(iii) the cumulative distribution function of **X**.

(iv) P(1 ≤ X ≤ 3)

[Ans: (i) $\frac{3}{16}$ (ii) 1.75 (iv) 0.6875]

50. The lifetime X in years of an electric bulb is a r.v **X** with the following p.d.f

$$f(x) = \begin{cases} \lambda x(5 - x), & 0 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the**:**

(i) value of λ**.**

(ii) mean of **X.**

(b) If two such new bulbs are sold, find the probability that**:**

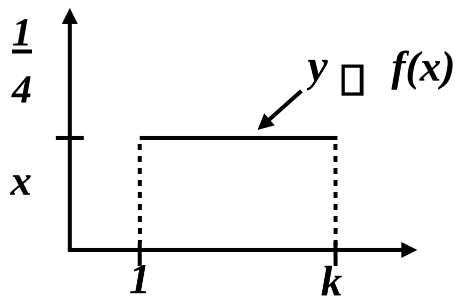
(i) both bulbs fail to work within one year.

(ii) only one bulb works for more than three years.

[Ans: (a) (i) $\frac{6}{125}$ (ii) 2.5 (b) (i) 0.0108 (ii) 0.4562]

51. A uniformly distributed **r.v** ***X*** has the following p.d.f ***f (x):***

y



Find the**:**

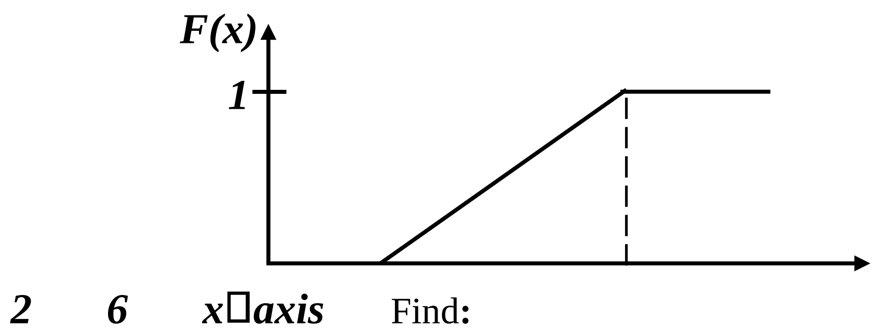
(i) value of ***k***

(ii) equations of the p.d.f of ***X***

(iii) variance of ***X***

[Ans: (i) 5 (iii) $\frac{4}{3}$]

52. The cumulative distribution function of a continuous ***r.v X*** is illustrated as follows:



Find**:**

(i) the p.d.f of **X** and sketch it**.**

(ii) the mean and variance of **X.**

(iii) $\mathbf{P(X > 3 \,/\, X < 5)}$

[Ans: (ii) 4, $\frac{4}{3}$ (iii) $\frac{2}{3}$]